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A LOWER BOUND FOR THE SPECTRAL RADIUS

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ABSTRACT. We prove an inequality for a problem of Carathéodory type: given n inner functions $m_1, m_2, \ldots, m_n$, to find the smallest norm of an H^∞ function such that the first n terms of its power series coincide with those of the product $m_1 \cdots m_n$. As an application, we obtain a lower bound for the spectral radius of an n-dimensional operator on Hilbert space in terms of its norm and the norm of its nth power.

In 1960 J. Mařík and the present author proved [4] that a linear operator A on n-dimensional l^∞ space which satisfies $|A| = |A^q| = 1$ for some $q > n^2 - n + 1$ has an eigenvalue of modulus one. Later [6] the present author introduced, for each Banach space E, the critical exponent of E as the smallest natural number q for which the following implication holds. If A is a linear operator on E and $|A| = |A^q| = 1$ then $|A|_\sigma = 1$. (We denote by $|A|$ the norm of A as a linear operator on E and by $|A|_\sigma$ its spectral radius.) In the same paper [6] the author proved that the critical exponent of Hilbert space equals its dimension. This result says the following. Consider a linear operator A in n-dimensional Hilbert space with $|A| = 1$; if $|A^n|$ is still one then A^m does not tend to zero as $m \to \infty$. In other words, if $|A| = 1$ and if the powers converge to zero then they start converging with the nth term at the latest. This, of course, is only a qualitative result. What would be of more use, however, is a quantitative version of it. If the convergence is still bad at the critical exponent q (i.e. if $|A^q|$ is still large) then it will continue being bad since we can infer from the behaviour of the first q terms that the spectral radius is close to one. To measure exactly how close to one, the author raised, in the next paper of the series [8], the following problem. Given a Banach space E, $0 < p < 1$, and a natural number r compute $\sup|A^r|$ where A ranges over all linear operators on E with $|A| = 1$ and $|A|_\sigma \leqslant p$. A simple compactness argument shows that this supremum $C(E, p, z^r)$ is < 1 for each $p < 1$ provided r is the critical exponent of E or some greater number.

In general, we can define, for each Banach space E, $0 < p < 1$, and each f holomorphic in a neighbourhood of the disc $|z| \leqslant p$,

$$C(E, p, f) = \sup\{|f(A)|;\ |A| \leqslant 1,\ |A|_\sigma \leqslant p\}.$$

In the same paper [8] this supremum was computed for the particular case of n-dimensional Hilbert space and the nth power–computed in the sense that an

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operator W was constructed for which $|W^n|$ is a maximum among all contractions on n-dimensional Hilbert space whose spectral radius does not exceed p.

With respect to a suitable basis **[10]** the matrix of W assumes a fairly simple form so that the nth power may be computed. The result is as follows. Consider a positive $p < 1$ and a natural number n. Let m be the function (holomorphic in a neighbourhood of the closed unit disc) defined as

$$m(z) = (z + p)/(1 + pz).$$

Denote by $c_0, c_1, c_2, \ldots$ the coefficients of the Taylor expansion $m(z)^n = c_0 + c_1 z + c_2 z^2 + \cdots$. If we write, for shortness, $C_n(p) = C(H_n, p, z^n)$ then $C_n(p) = |T|$ where T is the Toeplitz matrix

$$\begin{bmatrix} c_0 & 0 & 0 & \cdots & 0 \\ c_1 & c_0 & 0 & \cdots & 0 \\ c_2 & c_1 & c_0 & \cdots & 0 \\ \cdots & & & & \\ c_{n-1} & c_{n-2} & c_{n-3} & \cdots & c_0 \end{bmatrix}.$$

In the present paper these results will be used to determine the asymptotic behaviour of $C(H_n, p, z^n)$ as p tends to zero. The main theorem is based on a simple inequality for inner functions which seems to be of independent interest. This inequality is related to the classical Carathéodory problem and may be stated as follows. Given n inner functions $m_1, \ldots, m_n$ then there exists a function holomorphic in a neighbourhood of the unit disc such that the first n terms of its development coincide with those of the product $m_1 \cdots m_n$ and such that its maximum on the closed unit disc does not exceed $\sum_{i=1}^n |m_i(0)|$.

Notation. If E is a Banach space we denote by $\mathfrak{L}(E)$ the algebra of all bounded linear operators on E with the operator norm. We denote by H_n the n-dimensional Hilbert space, by C^n the concrete Hilbert space of n-dimensional column vectors with the l^2 norm. The unit vectors in C^n will be denoted by $e_1, \ldots, e_n$. By H^2 and H^∞ we denote the Hardy spaces, the corresponding norms will be written as $|\cdot|_2$ and $|\cdot|_\infty$. If $f \in H^\infty$ we define Sf to be the function

$$S(f) = (f(z) - f(0))/z.$$

The operator S is a contraction in H^2.

1. Products of inner functions. This section is devoted to the proof of an inequality for products of inner functions which forms the basis of the estimate of $C_n(p)$. The inequality seems to be interesting in its own right.

If $f \in H^\infty$ and n is a given natural number we shall denote by $\beta_n(f)$ the norm of f in the quotient space $H^\infty/z^n H^\infty$,

$$\beta_n(f) = \inf_{k \in H^\infty} |f - z^n k|_\infty.$$

Let us remark here that it can be shown that the infimum is attained.

(1.1) PROPOSITION. *Let $m_1, \ldots, m_n$ be given inner functions. Then there exists a $k \in H^\infty$ such that*

$$|m_1 \cdots m_n - z^n k|_\infty < \sum_{i=1}^{n} |m_i(0)|.$$

PROOF. By induction. Since $m_1 - zS(m_1) = m_1(0)$ we can set $k_1 = S(m_1)$. Suppose that $n > 1$ and that $k_n \in H^\infty$ is such that the function $f_n = m_1 \cdots m_n - z^n k_n$ has H^∞ norm $|f_n|_\infty < \sum_{i=1}^{n} |m_i(0)|$. Then

$$\begin{aligned} m_1 \cdots m_n m_{n+1} - z^{n+1} S(m_{n+1} k_n) \\ &= (f_n + z^n k_n) m_{n+1} - z^n (m_{n+1} k_n - m_{n+1}(0) k_n(0)) \\ &= f_n m_{n+1} + z^n m_{n+1}(0) k_n(0) \end{aligned}$$

so that, setting $k_{n+1} = S(m_{n+1} k_n)$, the following estimate holds:

$$|m_1 \cdots m_{n+1} - z^{n+1} k_{n+1}|_\infty < |f_n|_\infty + |m_{n+1}(0)|\, |k_n(0)|.$$

It follows that the induction will be complete if we prove that $|k_n(0)| < 1$. Since $|k_n(0)| < |k_n|_2$ this will be a consequence of the following stronger assertion: $|k_n|_2 < 1$ for all n. Now $|k_1|_2 = |S(m_1)|_2 < |m_1|_2 = 1$ and $|k_{n+1}|_2 = |S(m_{n+1} k_n)|_2 < |m_{n+1} k_n|_2 < |k_n|_2$ so that the proof is complete.

The preceding proposition may be reformulated as follows.

Given n inner functions $m_1, \ldots, m_n$ then

$$\beta_n(m_1 \cdots m_n) < \sum_{1}^{n} |m_j(0)|.$$

A closer look at the proof reveals the fact that equality is never attained here except in very special cases.

First of all, if $n = 1$, we have $\beta_1(m) = |m(0)|$ for every inner function m. Indeed, given $k \in H^\infty$, we have $|m(0)| = |(m - zk)(0)| < |m - zk|_\infty$ so that $|m(0)| < \beta_1(m)$. At the same time, for $k = S(m)$, we have $m - zk = m(0)$ whence $\beta_1(m) < |m(0)|$.

Thus for one function, we always have equality. The following result shows that, for $n > 1$, we have strict inequality almost always.

(1.2) PROPOSITION. *If $n > 1$ then*

$$\beta_n(m_1 \cdots m_n) = \sum_{1}^{n} |m_j(0)|$$

if and only if one of the following situations is obtained.

1°. *All $m_j(0)$ are zero.*

2°. *Exactly one $m_j(0)$ is different from zero all other m_q being of the form $\varepsilon_q z$ with $|\varepsilon_q| = 1$.*

PROOF. If all $m_j(0)$ are zero there exist inner functions a_j such that $m_j(z) = z a_j(z)$. Setting $k = a_1 \cdots a_n$ we have $m_1 \cdots m_n - z^n k = 0$ so that $\beta_n(m_1, \ldots, m_n)$ is zero and so is $\sum |m_j(0)|$.

Suppose now that $m_j(z) = z^{e_j}h_j(z)$ with $e_j \geqslant 1$ for $j = 1, 2, \ldots, n-1$ and that $m_n(0) \neq 0$. Set $w = e_1 + \cdots + e_{n-1}$ and $h = h_1 \cdots h_{n-1}$ so that $w \geqslant n-1$. We then have

$$m_1 \cdot \cdots \cdot m_n - z^n k = z^w h m_n - z^n k = z^{n-1}(z^{w-n+1}hm_n - zk).$$

Let us distinguish the following cases:

If $w = n-1$ then $|m_1 \cdots m_n - z^n k|_\infty = |hm_n - zk|_\infty$ so that $\beta_n(m_1 \cdots m_n) = \beta_1(hm_n) = |h(0)m_n(0)|$.

If h is constant then

$$\beta_n(m_1 \cdots m_n) = |m_n(0)| = \sum |m_j(0)|.$$

If h is nonconstant we have $|h(0)| < 1$ so that

$$\beta_n(m_1 \cdots m_n) < |m_n(0)| = \sum |m_j(0)|.$$

If $w \geqslant n$ we have $m_1 \cdots m_n - z^n k = z^n(z^{w-n}hm_n - k)$ so that

$$\beta_n(m_1 \cdots m_n) = 0 < \sum |m_j(0)|.$$

We see thus that, in the case that exactly one factor is not divisible by z, we always have sharp inequality except when all divisible m_j are of the form $\varepsilon_q z$ with $|\varepsilon_q| = 1$.

The proof will be complete if we show that we have sharp inequality if at least two $m_j(0)$ are different from zero. Hence suppose that $m_1(0) \neq 0$ and $m_2(0) \neq 0$. If m_1 is constant we have, for $k = 0$,

$$\beta_2(m_1m_2) \leqslant |m_1m_2|_\infty = 1 < 1 + |m_2(0)| = |m_1(0)| + |m_2(0)|.$$

If both m_1 and m_2 are nonconstant, we have, setting $k = S(S(m_1)m_2)$,

$$m_1(z)m_2(z) - z^2k(z) = m_1(0)m_2(z) + zm_1'(0)m_2(0)$$

whence $\beta_2(m_1m_2) \leqslant |m_1(0)| + |m_1'(0)|\,|m_2(0)|$.

Since $|m_1|_2 \leqslant 1$ and $m_1(0) \neq 0$ we have $|m_1'(0)| < 1$. It follows from this and from $m_2(0) \neq 0$ that $\beta_2(m_1m_2) < |m_1(0)| + |m_2(0)|$.

The rest follows from an inequality which we have already used.

In fact, since in each case we have found $k \in H^\infty$ with $|m_1m_2 - z^2k|_\infty < |m_1(0)| + |m_2(0)|$ and $|k|_2 \leqslant 1$, we may continue the construction used to prove Proposition (1.1) and conclude that

$$\beta_n(m_1 \cdots m_n) < |m_1(0)| + |m_2(0)| + \cdots + |m_n(0)|.$$

2. Asymptotics of $C_n(p)$. We shall assume $n > 1$ since the case $n = 1$ presents little interest.

The asymptotic behaviour of $C_n(p)$ as p tends to zero is completely described by the following

(2.1) PROPOSITION. *For* $0 < p < 1$,

$$np > C_n(p) \geqslant np(1 - p^2g(p))$$

where $\lim g(p) = n(n-1)/2$ *as* p *tends to zero.*

PROOF. The lower estimate is a consequence of the characterization of the extremal operator for $C_n(p)$. Indeed, we have

$$C_n(p) = |T_n(p)| \geqslant (T_n(p)e_n, e_1) = c_{n-1}$$
$$= \sum_{t=1}^{n-1} (-1)^{t-1}\binom{n}{t}\binom{n-2}{t-1}p^{2t-1}(1-p^2)^{n-t}$$

which yields the desired inequality for small p. The formula for c_{n-1} is obtained after a little calculation with the coefficients of $(z+p)/(1+pz)$.

Let us prove now, for all $0 < p < 1$, the inequality $C_n(p) < np$. To this end, consider the shift matrix

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & & \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and observe that $T_n(p) = m^n(E)$. By Proposition (1.1) there exists a $k_n \in H^\infty$ such that the function $f_n = m^n - z^n k_n$ has norm $|f_n|_\infty < n|m(0)| = np$. Since $E^n = 0$ we have $m^n(E) = f_n(E)$. By the von Neumann inequality $C_n(p) = |T_n(p)| = |m^n(E)| = |f_n(E)| \leqslant |f_n|_\infty < np$. This completes the proof.

(2.2) COROLLARY. *Let A be a nonzero linear operator on n-dimensional Hilbert space. Then*

$$n^{-1}|A|^{1-n}|A^n| \leqslant |A|_\sigma.$$

The coefficient n^{-1} is the best possible: If α satisfies $\alpha|A|^{1-n}|A^n| \leqslant |A|_\sigma$ for all nonzero $A \in B(H_n)$ then $\alpha \leqslant n^{-1}$.

The inequality is strict for $n > 1$.

PROOF. Set $p = |A|^{-1}|A|_\sigma$ so that $0 \leqslant p \leqslant 1$. The cases $p = 0$ and $p = 1$ being easy, let us assume $0 < p < 1$. Set $W = |A|^{-1}A$ so that $|W| = 1$ and $|W|_\sigma = p$. Then

$$|A|^{1-n}|A^n| = |A|\,|W^n| \leqslant |A|C_n(p) \leqslant |A|n|A|^{-1}|A|_\sigma = n|A|_\sigma$$

whence the assertion. That no greater constant than n^{-1} will do is a consequence of the other inequality in Proposition (2.1).

ADDED IN PROOF. Following a lecture given by the author about this matter in Bucarest, C. Apostol found an inequality for the trace norm of an operator from which $C_n(p) \leqslant np$ can also be deduced. This inequality is reproduced in the survey paper V. Pták and N. J. Young, *Functions of operators and the spectral radius* accepted by Linear Algebra and its Applications.

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