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ON ORDERED HARMONIC BOUNDED VARIATION

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ABSTRACT. An example is given of a continuous real function that is of ordered harmonic bounded variation but not of harmonic bounded variation.

Let f be a real-valued function defined on [0, 1], and for each open interval $I = (a, b) \subset [0, 1]$ set $f(I) = f(b) - f(a)$. Then f is said to be of *harmonic bounded variation* (HBV) on [0, 1] if

$$\sup \sum |f(I_n)|/n < \infty, \tag{†}$$

where the supremum is taken over all sequences of nonoverlapping open intervals $I_1, I_2, \ldots$ in [0, 1]; if (†) holds when the supremum is taken over all sequences of open intervals $I_1, I_2, \ldots$ in [0, 1] for which either $I_j < I_{j+1}$ for each index j or $I_{j+1} < I_j$ for each index j (where $I < J$ means I lies to the left of J), then f is said to be of *ordered harmonic bounded variation* ($OHBV$) on [0, 1]. These two function classes were introduced by D. Waterman in [1] and [2], respectively, and in [2] he asked whether the inclusion $HBV \subset OHBV$ is proper. Here we show that it is.

LEMMA. *Let N be a nonnegative integer, M a positive integer, and* $0 < \varepsilon < 1$. *Then there exists an integer $T > M$ and a corresponding sequence of positive numbers $A_{N+1} \geqslant A_{N+2} \geqslant \cdots \geqslant A_{N+T}$ for which the following hold*:

(i) $\sum_{j=1}^{M} A_{N+j}/j < \varepsilon$,

(ii) $\sum_{j=1}^{T} A_{N+j}/(N+j) > 1$ *but* $\sum_{j=1}^{T} A_{N+j}/j < 4$,

(iii) $\sum_{j=1}^{t} A_{N+t+1-j}/j < 6\varepsilon$ $(1 \leqslant t \leqslant T)$.

PROOF. Define $A_{N+1} = A_{N+2} = \cdots = A_{N+M} = \varepsilon/(N+M+1)$ and

$$A_j = A_{N+1}/k \quad \text{for } (N+M)2^{k-1} < j \leqslant (N+M)2^k \ (k \geqslant 1).$$

Noting that (i) is satisfied, we proceed to prove (ii).

First observe that if $N + T = (N+M)2^K$ $(K \geqslant 1)$, then

$$\sum_{j=1}^{T} A_{N+j}/(N+j)$$
$$= A_{N+1}\left\{\sum_{j=1}^{M} 1/(N+j) + \sum_{k=1}^{K}\left(\frac{1}{k}\right)\Sigma\left((N+M)2^{k-1}, (N+M)2^k\right]\right\}, \tag{*}$$

where we use the notation $\Sigma(a, b] = \sum_{j=a+1}^{b} 1/j$. For $K = 1$, the right-hand side of (*) is $\leqslant A_{N+1}(M+1)$, which by the definition of A_{N+1} is < 1. Also, since

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$$\ln 2 > \sum (n, 2n] \to \ln 2 \quad \text{as } n \to \infty,$$

the second series on the right-hand side of (*) is a partial sum of a divergent series with its kth term ($k > 1$) less than $(\ln 2)/2$; hence, for some T we have

$$1 < \sum_{j=1}^{T} A_{N+j}/(N+j) < 2, \tag{**}$$

which proves the first part of (ii). The second part follows from (**) and the simple observation that

$$\sum_{j=1}^{T} A_{N+j}/j - \sum_{j=1}^{T} A_{N+j}/(N+j) = \sum_{j=1}^{T} NA_{N+j}/j(N+j) < \sum_{j=1}^{T} \varepsilon/j(N+j) < 2.$$

To begin consideration of (iii) we note that $A_{N+t+1-j} = A_{N+j}$ for $j \leqslant t \leqslant M$, and hence, that (iii) for $1 \leqslant t \leqslant M$ follows from (i).

Now suppose t satisfies $(N+M)2^{K-1} < N + t \leqslant (N+M)2^K$ for some $K \geqslant 1$. Then let S_n ($n = 1, 2, \ldots, K$) denote the sum

$$\sum_j A_{N+t+1-j}/j \quad \text{for } (N+M)2^{K-n} < N + t + 1 - j \leqslant (N+M)2^{K-n+1},$$

and let S_{K+1} denote this sum for $N + 1 \leqslant N + t + 1 - j \leqslant N + M$.

Consider S_1. For the indices j involved in this sum, we have $A_{N+t+1-j} = A_{N+1}/K$ and hence

$$\begin{aligned} S_1 &\leqslant (A_{N+1}/K)\Big(1 + \sum \big(1, (N+M)2^{K-1}\big]\Big) \\ &< \big[\varepsilon/K(N+M+1)\big]\big[1 + \ln(N+M)2^{K-1}\big] < 2\varepsilon. \end{aligned}$$

Similarly $S_2 < 2\varepsilon$.

Now consider S_n for $3 \leqslant n \leqslant K$. Since $A_{N+t+1-j} = A_{N+1}/(K-n+1)$ for each of the $(N+M)2^{K-n}$ indices j involved in S_n and since the first of these indices is greater than $(N+M)2^{K-2}$, which is the number of terms in S_2, we see that

$$\begin{aligned} S_n &< \big[A_{N+1}/(K-n+1)\big]\big[(N+M)2^{K-n}/(N+M)2^{K-2}\big] \\ &= \big[A_{N+1}/(K-n+1)\big]/2^{n-2}. \end{aligned}$$

Then, since $A_{N+1} < \varepsilon$, it follows that $S_n < \varepsilon/2^{n-2}$ for $3 \leqslant n \leqslant K$.

Combining these estimates with the observation that S_{K+1} is dominated by $\sum_{j=1}^{M} A_{N+j}/j$ which by (i) is $< \varepsilon$, we have

$$\sum_{j=1}^{t} A_{N+t+1-j}/j < 2\varepsilon + 2\varepsilon + \varepsilon \sum_{n=3}^{K} 2^{2-n} + \varepsilon < 6\varepsilon,$$

and the lemma is proved.

THEOREM. *There exists a continuous $f \in OHBV - HBV$.*

PROOF. Choose positive numbers $\varepsilon_1, \varepsilon_2, \ldots$ so that $\sum_{n=1}^{\infty} \varepsilon_n = 1$. Then apply the Lemma for $N = 0$, $M = M_1 \equiv 1$, $\varepsilon = \varepsilon_1$ to obtain an integer $T_1 > 1$ and a

sequence of positive numbers $A_1 \geqslant A_2 \geqslant \cdots \geqslant A_{T_1}$ such that

(i) $\sum_{j=1}^{M_1} A_j/j < \varepsilon_1$,

(ii) $1 < \sum_{j=1}^{T_1} A_j/j < 4$,

(iii) $\sum_{j=1}^{t} A_{t+1-j}/j < 6\varepsilon_1$ $(1 \leqslant t \leqslant T_1)$.

Now choose $M_2 > T_1$ so large that

$$\sum_{j=1}^{T_1} A_j/(M_2 + j) < \varepsilon_1.$$

Then apply the Lemma for $N = T_1$, $M = M_2$, $\varepsilon = \varepsilon_2$ to obtain an integer $T_2 > M_2$ and a sequence of positive numbers $A_{T_1+1} \geqslant A_{T_1+2} \geqslant \cdots \geqslant A_{T_1+T_2}$ such that

(i) $\sum_{j=1}^{M_2} A_{T_1+j}/j < \varepsilon_2$,

(ii) $1 < \sum_{j=1}^{T_2} A_{T_1+j}/(T_1 + j)$ but $\sum_{j=1}^{T_2} A_{T_1+j}/j < 4$,

(iii) $\sum_{j=1}^{t} A_{T_1+t+1-j}/j < 6\varepsilon_2$ $(1 \leqslant t \leqslant T_2)$.

Now choose $M_3 > T_2$ so large that

$$\sum_{j=1}^{T_2} A_{T_1+j}/(M_3 + j) < \varepsilon_2.$$

Then apply the Lemma for $N = T_1 + T_2$, $M = M_3$, $\varepsilon = \varepsilon_3$ and continue inductively to obtain: two sequences of positive integers $M_1, M_2, \ldots$ and $T_1, T_2, \ldots$ satisfying

$$M_k < T_k < M_{k+1} \quad (k \geqslant 1),$$

and a sequence of positive numbers $A_1, A_2, \ldots$ satisfying

$$A_{\sigma_k+1} \geqslant A_{\sigma_k+2} \geqslant \cdots \geqslant A_{\sigma_k+T_k} \quad \left(\sigma_k \equiv \sum_{j=0}^{k-1} T_j;\ T_0 \equiv 0;\ k \geqslant 1\right)$$

such that

(a) $\sum_{j=1}^{M_k} A_{\sigma_k+j}/j < \varepsilon_k$,

(b) $\sum_{j=1}^{T_k} A_{\sigma_k+j}/(\sigma_k + j) > 1$ but $\sum_{j=1}^{T_k} A_{\sigma_k+j}/j < 4$,

(c) $\sum_{j=1}^{T_k} A_{\sigma_k+j}/(M_{k+1} + j) < \varepsilon_k$,

(d) $\sum_{j=1}^{t} A_{\sigma_k+t+1-j}/j < 6\varepsilon_k$ $(1 \leqslant t \leqslant T_k)$.

Now we proceed to define the desired function. Inductively we obtain a sequence of mutually disjoint closed intervals $I_n \equiv [a_n, b_n]$ contained in $(0, 1)$ such that for each index k we have

$$I_i < I_{\sigma_k+j} < I_{\sigma_k+j+1} \qquad (j = 1, 2, \ldots, T_k - 1;\ i > \sigma_{k+1}).$$

Then we define $f(x) = 0$ if $x \notin (a_n, b_n)$ for each n, $f((a_n + b_n)/2) = A_n$, and linearly extend f to the remainder of $[0, 1]$.

For each index n set $I_n^* = (a_n, (a_n + b_n)/2)$ and note that, by the first inequality in (b), we have

$$\sum_{n=1}^{\infty} f(I_n^*)/n = \sum_{n=1}^{\infty} A_n/n = \sum_{k=1}^{\infty} \sum_{j=1}^{T_k} A_{\sigma_k+j}/(\sigma_k + j) = \infty;$$

that is, $f \notin HBV$.

To show that $f \in OHBV$, it suffices to prove the finiteness of the supremum of the sums $\sum f(J_n)/n$ taken over all ordered sequences of intervals $J_1, J_2, \ldots$ from the collection $\{I_n^*\}_{n=1}^{\infty}$.

First suppose $J_1 > J_2 > \cdots$. Then it follows readily from (d) that

$$\sum f(J_n)/n < 6 \sum_{k=1}^{\infty} \varepsilon_k = 6.$$

Now suppose $J_1 < J_2 < \cdots$. Choose an integer k_0 and let J_{n_0+1}, $J_{n_0+2}, \ldots, J_{n_0+t_0}$ be the J_n's that equal one of the intervals $I^*_{\sigma_{k_0}+j}$ ($j = 1, 2, \ldots, T_{k_0}$). Since $A_{\sigma_{k_0}+j} \geqslant A_{\sigma_{k_0}+j+1}$ ($j = 1, 2, \ldots, T_{k_0} - 1$), we have

$$\Sigma_0 \equiv \sum_{j=1}^{t_0} f(J_{n_0+j})/(n_0 + j) \leqslant \sum_{j=1}^{t_0} A_{\sigma_{k_0}+j}/j.$$

Hence, if $t_0 \leqslant M_{k_0}$, it follows from (a) that $\Sigma_0 < \varepsilon_{k_0}$. Now consider the case $t_0 > M_{k_0}$: By the second inequality in (b), we have $\Sigma_0 < 4$; furthermore, if $J_{n_0+t_0+i}$ ($p \leqslant i \leqslant q$) are the J_n's that equal one of the intervals $I^*_{\sigma_k+j}$ ($j = 1, 2, \ldots, T_k$) for a fixed $k < k_0$, then since $A_{\sigma_k+j} \geqslant A_{\sigma_k+j+1}$ ($j = 1, 2, \ldots, T_k$) we have

$$\sum_{i=p}^{q} f(J_{n_0+t_0+i})/(n_0 + t_0 + i) < \sum_{j=1}^{q-p+1} A_{\sigma_k+j}/(M_{k_0} + j) < \varepsilon_k,$$

where the last inequality follows from (c) and the fact that $M_{k_0} \geqslant M_{k+1}$ for $k < k_0$. Consequently,

$$\sum f(J_n)/n < 4 + \sum_{k=1}^{\infty} \varepsilon_k = 5,$$

and $f \in OHBV$.

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References

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