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INSCRIBED AND CIRCUMSCRIBED CIRCLES TO CONVEX CURVES

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ABSTRACT. A convex planar curve may have 2, 3, ..., $\mathfrak{c}$ contact points with its inscribed or circumscribed circle. One of these numbers appears in most cases: 3.

Let $\mathcal{C}$ be the space of all closed convex curves in the plane (see [1, p. 3] for a precise definition of a closed convex surface, in particular curve). Several pathological properties of most curves in $\mathcal{C}$ (in the sense of Baire categories) are described in [4] and [5]. We shall show here that most curves in $\mathcal{C}$ have the (expected?) number, 3, of contact points with their inscribed and circumscribed circles. It seems that mainly local properties may be pathological for most curves in $\mathcal{C}$.

We say that *most* elements of a space of second Baire category have a certain property if those elements which do not have it form a set of first Baire category. Now, $\mathcal{C}$ is of second Baire category (see for instance [5]) if we endow it with the Hausdorff metric, so it makes sense to speak about most curves in $\mathcal{C}$.

Let $C \in \mathcal{C}$, D be the convex domain with boundary C, K_C the *circumscribed circle* of C, i.e. the smallest circle surrounding D, and k_C an *inscribed circle* of C, i.e. a largest circle included in $D \cup C$. The circle k_C is not unique only if C contains parallel segments. Since most curves in $\mathcal{C}$ are strictly convex (see [3] or [2]), they admit a unique inscribed circle.

Clearly, $\operatorname{card}(C \cap K_C)$ may be any cardinal number between 2 and $\mathfrak{c}$. We prove

THEOREM 1. *For most curves* $C \in \mathcal{C}$, $\operatorname{card}(C \cap K_C) = 3$.

We will use the following elementary Lemma, that we give without proof.

LEMMA. *Let P be a polygon such that $P \cap K_P$ consists of precisely three points x_1, x_2, x_3 determining an acute triangle. Let N_1, N_2, N_3 be neighborhoods of x_1, x_2, x_3. Then there is a neighborhood $\mathfrak{N}$ of P in $\mathcal{C}$ such that, for each $C \in \mathfrak{N}$,*

$$C \cap K_C \cap N_i \neq \varnothing \qquad (i = 1, 2, 3)$$

and

$$C \cap K_C \subset N_1 \cup N_2 \cup N_3.$$

PROOF OF THEOREM 1. We first show that the set $\mathcal{C}_2$ of all curves in $\mathcal{C}$ satisfying $\operatorname{card}(C \cap K_C) = 2$ is nowhere dense in $\mathcal{C}$.

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Let $\mathcal{O}$ be an open set in $\mathcal{C}$. We choose a polygon P in $\mathcal{O}$.

If $\operatorname{card}(P \cap K_P) = 2$, let x and y be the vertices of P lying on K_P. Let x_1, x_2 be two points such that the segment x_1x_2 contains x and is orthogonal on xy. Let K' be the circle through x_1, x_2, y. The boundary P' of the convex hull of $P \cup \{x_1, x_2\}$ has K' as circumscribed circle. If x_1 and x_2 are close enough to x, P' still lies in $\mathcal{O}$. Obviously

$$P' \cap K' = \{x_1, x_2, y\}.$$

If $\operatorname{card}(P \cap K_P) \geqslant 3$, there are three points x_1, x_2, x_3 in $P \cap K_P$ determining a triangle with all angles of measure at most $\pi/2$. By gently cutting all the other vertices of P and slightly moving x_1 if necessary, we obtain a polygon P' still belonging to $\mathcal{O}$ such that $P' \cap K_{P'}$ is the vertex-set of an acute triangle. Now, by the Lemma, for a neighborhood $\mathfrak{N}$ of P' in $\mathcal{C}$, each curve $C \in \mathfrak{N}$ meets K_C in at least three points. Thus

$$\mathcal{O} \cap \mathfrak{N} \cap \mathcal{C}_2 = \varnothing,$$

which proves that $\mathcal{C}_2$ is nowhere dense in $\mathcal{C}$.

Let $\mathcal{C}_{(n)}$ be the set of all curves C in $\mathcal{C}$ such that

(i) $\operatorname{card}(C \cap K_C) \geqslant 4$, and

(ii) there exist four points $x_1, x_2, x_3, x_4 \in C \cap K_C$ such that the side-lengths of the convex quadrangle with vertices x_1, x_2, x_3, x_4 are at least n^{-1} ($n \in \mathbf{N}$).

We show that $\mathcal{C}_{(n)}$ is nowhere dense in $\mathcal{C}$.

Let $\mathcal{O}$ be an open set in $\mathcal{C}$. We choose like before a polygon P' in $\mathcal{O}$ such that $P' \cap K_{P'}$ is the vertex-set of an acute triangle. Now let N_i be a disk of centre x_i and radius less than $(2n)^{-1}$. By the Lemma, there is a neighborhood $\mathfrak{N}$ of P' such that, for each curve $C \in \mathfrak{N}$,

$$C \cap K_C \subset N_1 \cup N_2 \cup N_3,$$

and therefore we cannot find 4 points in $C \cap K_C$ determining a convex quadrangle with side-lengths at least n^{-1}. Thus

$$\mathcal{O} \cap \mathfrak{N} \cap \mathcal{C}_{(n)} = \varnothing$$

and $\mathcal{C}_{(n)}$ is nowhere dense in $\mathcal{C}$.

Let $\mathcal{C}_3$ be the set of all curves $C \in \mathcal{C}$ verifying $\operatorname{card}(C \cap K_C) = 3$. Every curve of $\mathcal{C}$ not belonging to $\mathcal{C}_2$ or $\mathcal{C}_3$ must be in $\mathcal{C}_{(n)}$ for some $n \in \mathbf{N}$. Thus

$$\mathcal{C} - \mathcal{C}_3 = \mathcal{C}_2 \cup \bigcup_{n=1}^{\infty} \mathcal{C}_{(n)},$$

where $\mathcal{C}_2$ and $\mathcal{C}_{(n)}$ ($n = 1, 2, 3, \dots$) are nowhere dense; therefore $\mathcal{C} - \mathcal{C}_3$ is of first Baire category, which proves the theorem.

Surprisingly enough, the proof of Theorem 2 which follows is so similar to the preceding one, that we do not need to give it separately.

Like in the case of K_C, $C \cap k_C$ may be any cardinal number between 2 and $\mathfrak{c}$.

THEOREM 2. *For most curves* $C \in \mathcal{C}$, $\operatorname{card}(C \cap k_C) = 3$.

The above results extend to higher dimensions. Since no technical difficulties appear in connection with increased dimension, we choose to present the planar case as a typical one.

Let $\mathbb{S}^d$ be the space of all (d-dimensional) closed convex surfaces S in $\mathbf{R}^{d+1}$. Let K_S and k_S be the circumscribed and an inscribed hypersphere of $S \in \mathbb{S}^d$.

THEOREM 3. *For most surfaces* $S \in \mathbb{S}^d$,

$$\operatorname{card}(S \cap K_S) = \operatorname{card}(S \cap k_S) = d + 2.$$

The proof parallels that of Theorems 1 and 2.

REFERENCES

1. H. Busemann, *Convex surfaces*, Interscience, New York, 1958.
2. P. Gruber, *Die meisten konvexen Körper sind glatt, aber nicht zu glatt*, Math. Ann. **229** (1977), 259–266.
3. V. Klee, *Some new results on smoothness and rotundity in normed linear spaces*, Math. Ann. **139** (1959), 51–63.
4. R. Schneider, *On the curvatures of convex bodies*, Math. Ann. **240** (1979), 177–181.
5. T. Zamfirescu, *The curvature of most convex surfaces vanishes almost everywhere* (to appear).

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