

A MANY-SORTED INTERPOLATION THEOREM FOR $L(Q)$

DAVID R. GUICHARD

ABSTRACT. Let L be a many-sorted relational language with \in and consider the logic $L_{\omega_1\omega}(Q)$, infinitary logic with a monotone quantifier. We prove a version of Feferman's Interpolation Theorem for this logic. We then use the theorem to show that for a one-sorted language L and a countable admissible fragment L_A of $L_{\omega_1\omega}(Q)$, any sentence which persists for end extensions is equivalent to a Σ sentence.

1. Introduction. In a course on monotone quantifiers and admissible sets Jon Barwise raised the question of which sentences of $L_{\omega_1\omega}(Q)$ are preserved for end extensions. He subsequently suggested that a many-sorted interpolation theorem, à la Feferman, would be of interest in its own right and might provide a proof of a preservation theorem as well; this turned out to be the case. We are grateful to Barwise for his guidance and to Kim Bruce for his helpful comments. The interpolation theorem we prove generalizes a theorem of Shelah in [3].

2. Languages. We will be dealing with various fragments of $L_{\omega_1\omega}(Q)$, where L is a countable many-sorted language including \in and Q is a 'bounded quantifier' symbol. The notions of fragment, formula, proof, etc., are all completely analogous to $L_{\omega_1\omega}$ and to $L(Q)$. A brief description of many sorted $L_{\omega_1\omega}$ can be found in Nebres [6]; for bounded quantifiers see Barwise [2]. For the record, we note a few of the slight changes in these notions—just the obvious modifications to blend the ideas smoothly.

DEFINITION. Let L be a language and J the set of sorts of L . Then we say (\mathfrak{A}, q) is an $L(Q)$ structure if \mathfrak{A} is an L structure and q is a function with domain $J \times |\mathfrak{A}|$ such that $q(i, a)$, also written q_i^a , is a quantifier that lives on a_E (E is the interpretation of \in in the model \mathfrak{A}); we write $\check{q}(i, a)$ to denote $\{X: |\mathfrak{A}| - X \notin q(i, a)\}$, the dual of q .

Given a term t , we often write t^i to indicate that t is of sort $i \in J$. If u is also a term, we may say 'u is of sort t ' when we mean that u is of the same sort as t .

DEFINITION. Satisfaction is defined as expected. The clause for Q is $(\mathfrak{A}, q) \models Q^y x^i \phi$ iff $\{a^i \in \mathfrak{A} | (\mathfrak{A}, q) \models \phi[a]\} \in q_i^y$. \check{Q} is the dual of Q ; by the above definition we have $(\mathfrak{A}, q) \models \check{Q}^y x^i \phi$ iff $\{a^i \in \mathfrak{A} | (\mathfrak{A}, q) \models \phi[a]\} \in \check{q}_i^y$.

We axiomatize bounded quantifiers as follows. For each x, z of sort i and y of sort j :

Received by the editors August 7, 1979.

AMS (MOS) subject classifications (1970). Primary 02B20, 02B25.

© 1980 American Mathematical Society
 0002-9939/80/0000-0569/\$02.50

- A1. $Q^y x\phi \leftrightarrow Q^y z\phi$;
 A2. $Q^y x\phi \leftrightarrow \neg Q^y x \neg\phi$;
 A3. $\forall x (\phi(x, y) \rightarrow \psi(x, y)) \rightarrow (Q^y x\phi \rightarrow Q^y x\psi)$;
 A4. $Q^y x\phi \leftrightarrow Q^y x (x \in y \wedge \phi)$.

3. Consistency properties. The reader should be familiar with consistency properties, at least as they are used in the proofs of interpolation theorems. In particular, the proof of the Malitz Interpolation Theorem in Keisler [5] and the discussion of many-sorted consistency properties in [6] will be useful references. Our definition of consistency property differs from that of Nebres in the addition of C8 (to handle Q) and in the formulation of C4 in the presence of bounded quantifiers. We remark that we often write ' $s + \psi$ ' for ' $s \cup \{\psi\}$ ' when s is a set of formulas and ψ is a single formula. Recall that $\sim\phi$ denotes a formula equivalent to $\neg\phi$, obtained by pushing the negation 'one step' into ϕ . See [1, p. 84] for a precise definition.

DEFINITION. Let K be $L(C)_{\omega, \omega}(Q)$ where C contains \aleph_0 new constant symbols of each sort. A set S is a consistency property for K if each $s \in S$ is a countable set of sentences of K satisfying all the following conditions. We write a, c, d for elements of C , t for a term of $L(C)$.

- C0. $0 \in S$; $s' \subset s$ implies $s' \in S$.
 C1. $\phi \in s$ implies $\neg\phi \notin s$.
 C2. $\neg\phi \in s$ implies $s + \sim\phi \in S$.
 C3. $\bigwedge \Phi \in s$ implies $s + \phi \in S$ for each $\phi \in \Phi$.
 C4. (i) $\forall v \in a\phi \in s$ and $(c \in a) \in s$ and c of sort v implies $s + \phi(c) \in S$;
 (ii) $\forall v\phi \in s$ and c of sort v implies $s + \phi(c) \in S$.
 C5. $\bigvee \Phi \in s$ implies there is a $\phi \in \Phi$ with $s + \phi \in S$.
 C6. $\exists v\phi \in s$ implies that for some c of sort v , $s + \phi(c) \in S$.
 C7. $(c = d) \in s$ implies $s + (d = c) \in S$; $\phi(t), (t = c) \in S$ implies $s + \phi(c) \in S$; for each t there is a c such that $s + (c = t) \in S$.
 C8. If $Q^a x\phi$ and $\tilde{Q}^a x\psi$ are in s , then for some c of sort x , $s \cup \{(c \in a), \phi(c), \psi(c)\} \in S$.

We now prove the model existence theorem for this notion of consistency property; the proof is a straightforward modification of the proof in [5].

THEOREM. *Suppose S is a consistency property and $s_0 \in S$. Then s_0 has a model.*

PROOF. Let K_A be a countable fragment containing s_0 and all sentences of the form $(c = d)$ for c, d in C . List the sentences of K_A as $\{\phi_0, \phi_1, \phi_2, \dots\}$ and the terms as $\{t_0, t_1, t_2, \dots\}$. We construct a sequence $s_0 \subset s_1 \subset s_2 \dots$ of elements of S . Given s_n we may choose s'_n so that

- (1) $s_n \subset s'_n \in S$.
- (2) If $s_n + \phi_n \in S$, then $\phi_n \in s'_n$.
- (3) If $s_n + \phi_n \in S$ and ϕ_n is $\bigvee \Phi$, then for some $\theta \in \Phi$, $\theta \in s'_n$.
- (4) If $s_n + \phi_n \in S$ and ϕ_n is $\exists x\phi$, then for some c of sort x , $\phi(c) \in s'_n$.
- (5) For some $c \in C$, $(c = t_n) \in s'_n$.

Finally, choose $s_{n+1} \in S$ so that $s'_n \subset s_{n+1}$ and

(6) For each pair $\{Q^ax\psi, \check{Q}^ax\sigma\} \subset s'_n \cap \{\phi_0, \dots, \phi_n\}$, there is a $c \in C$ of sort x such that $\{\psi(c), \sigma(c), (c \in a)\} \subset s_{n+1}$.

Let $s_\omega = \cup s_n$. For c, d in C let $c \sim d$ iff $(c = d) \in s_\omega$. Then \sim is an equivalence relation; let $M = C/\sim$, and define the model \mathfrak{M} as usual. Define $q(j, [d])$ to be the monotone quantifier generated by all sets of the form $\{[c^j] \mid (c \in d), \psi(c) \text{ are in } s_\omega\}$ for each $Q^ax^i\psi \in s_\omega$. Now by induction we show that $\phi \in s_\omega$ implies that $\mathfrak{M} \models \phi$. The only new steps are the Q and \check{Q} cases; the bounded \forall is not any harder than the usual \forall step.

(i) Suppose $Q^ax^i\psi \in s_\omega$. Then $\{[c] \mid \psi(c), (c \in a) \in s_\omega\}$ is in $q(i, [a])$. By induction this is a subset of $\{[c] \mid \mathfrak{M} \models \psi(c) \wedge (c \in a)\}$, so $\mathfrak{M} \models Q^ax\psi$.

(ii) Suppose $\check{Q}^ax^i\psi \in s_\omega$. We need to show that $\{[c] \mid \mathfrak{M} \not\models \psi(c)\} \in \check{q}(i, [a])$. Suppose on the contrary that $U = \{[c] \mid \mathfrak{M} \not\models \psi(c)\} \in q(i, [a])$. Then U contains some set $V = \{[c] \mid (c \in a), \sigma(c) \in s_\omega\}$, where $Q^ax\sigma \in s_\omega$. Hence for some n , $\{Q^ax\psi, Q^ax\sigma\} \subset s_n$, and so there is a $c \in C$ such that $\{(c \in a), \psi(c), \sigma(c)\} \subset s_{n+1}$. But then $[c]$ is in $V - U$, a contradiction.

4. Interpolation theorem. After establishing some notation, we prove the analogue of Feferman's theorem. The notation is essentially the same as in [4]. Recall that ϕ^* is the negation normal form of ϕ .

DEFINITION. Suppose ϕ is a sentence of $L_{\omega_1\omega}(Q)$. Then $\text{Rel}(\phi)$ and $\text{Cn}(\phi)$ denote respectively the sets of relation and constant symbols occurring in ϕ . $\text{Sort}(\phi)$ is the set of $j \in J$ such that ϕ contains a term of sort j . Also

$$\text{Qu}(\phi) = \{j \in J \mid \text{a variable of sort } j \text{ appears bound by } Q \text{ or } \check{Q} \text{ in } \phi\}.$$

$$\text{Un}(\phi) = \{j \in J \mid \text{a variable of sort } j \text{ appears bound by } \forall \text{ in } \phi^*\}.$$

$$\text{Un}^*(\phi) = \{j \in J \mid \text{a variable of sort } j \text{ appears in an unbounded occurrence of } \forall \text{ in } \phi^*\}.$$

Define $\text{Ex}(\phi)$ and $\text{Ex}^*(\phi)$ similarly for \exists . Finally, if s is a set of sentences we write $\text{Rel}(s)$, $\text{Cn}(s)$, etc., for the appropriate unions over $\phi \in s$.

DEFINITION. Given sentences ϕ, ψ such that $\models \phi \rightarrow \psi$, we say that a sentence θ is an interpolant between ϕ and ψ if

- (1) $\models \phi \rightarrow \theta$ and $\models \theta \rightarrow \psi$.
- (2) $F(\theta) \subset F(\phi) \cap F(\psi)$ whenever F is Rel , Cn , Qu or Sort .
- (3) $\text{Un}(\theta) \subset \text{Un}(\phi)$, $\text{Un}^*(\theta) \subset \text{Un}^*(\phi)$, $\text{Ex}(\theta) \subset \text{Ex}(\psi)$ and $\text{Ex}^*(\theta) \subset \text{Ex}^*(\psi)$.

THEOREM. Suppose L has no function symbols, that ϕ, ψ are sentences of some countable admissible fragment L_A and $\models \phi \rightarrow \psi$. Then there is an interpolant $\theta \in L_A$ between ϕ and ψ .

The strategy for the proof is the usual one. We define a consistency property in an advantageous way and the theorem follows easily.

DEFINITION. Let L_A, ϕ, ψ be as above. Let $K_A = L_A(C)$, where C contains \aleph_0 new constant symbols of each sort. Let S be the set of all finite sets s of sentences of K_A such that only finitely many $c \in C$ occur in s and such that $s = s_1 \cup s_2$ so that

(*) There is no sentence θ of K_A satisfying all of the following.

- (1) $s_1 \models \theta$ and $s_2 \models \neg\theta$.
- (2) $F(\theta) \subset F(\phi) \cap F(\psi)$ when F is Rel, Cn, Sort or Qu.
- (3) $\text{Un}(\theta) \subset \text{Un}(s_1)$, $\text{Un}^*(\theta) \subset \text{Un}^*(s_1)$, $\text{Ex}(\theta) \subset \text{Un}(s_2)$, $\text{Ex}^*(\theta) \subset \text{Un}^*(s_2)$.

LEMMA. *The Barwise Completeness Theorem is true for countable admissible fragments $L_A(Q)$.*

PROOF. A direct modification of the ordinary proof will work; see [1].

LEMMA. *The set S defined above is a consistency property.*

PROOF. This is almost the same as in Barwise [1] and Keisler [5]. We will indicate the modifications needed. We must verify C8 which has been added here and C4 which has been altered to handle bounded \forall . We also need to treat C7 more carefully, since we have to keep a sharp eye on quantifiers. Note that the verification of C5 requires Barwise Completeness; see [1] for details.

C4: The only new argument occurs for the bounded quantifier. There are two interesting cases depending on whether $\forall v \in a\phi$ is in s_1 or in s_2 . Suppose it is in s_2 , while $(c \in a) \in s_1$ and $c \notin \text{Cn}(s_2)$. Then we claim s_1 and $s_2 + \phi(c)$ satisfy (*) so that $s + \phi(c) \in S$. If not, there is a θ such that $s_1 \models \theta(c)$ but $s_2 + \phi(c) \models \neg\theta(c)$. Then it is easy to see that $s_1 \models \exists x \in a\theta$ and $s_2 \models \neg\exists x \in a\theta$, which contradicts the fact that s_1 and s_2 satisfy (*). Other cases are similar or easier.

C7: The only tricky case is this: suppose $\sigma(t) \in s_1$, $(c = t) \in s_2$ and $c \notin \text{Cn}(s_1)$. Then we need to show that $s + \sigma(c) \in S$. It suffices to show that $s_1 + (c = t)$ and s_2 satisfy (*). Suppose not. Then there is a sentence $\theta(c)$ such that $s_1 + (c = t) \models \theta(c)$ and $s_2 \models \neg\theta(c)$. Then $s_1 \models \theta(t)$, $s_2 \models \neg\theta(t)$, a contradiction.

C8: There are as usual a number of cases; a representative one should suffice. Suppose $Q^a x\sigma$ is in s_1 , $\check{Q}^a x\pi$ is in s_2 . Pick $c \notin \text{Cn}(s)$. We claim $s_1 + (c \in a) \wedge \sigma(c)$, $s_2 + \pi(c)$ will satisfy (*). If not,

$$\begin{aligned} s_1 + (c \in a) \wedge \sigma(c) \models \theta(c), & \quad s_2 + \pi(c) \models \neg\theta(c); \\ s_1 \models \forall x(x \in a \wedge \sigma(x) \rightarrow \theta(x)), & \quad s_2 \models \forall x(\pi(x) \rightarrow \neg\theta(x)). \end{aligned}$$

By monotonicity (axiom A3) and A4,

$$\begin{aligned} s_1 \models Q^a x\sigma \rightarrow Q^a x\theta, & \quad s_2 \models \check{Q}^a x\pi \rightarrow \check{Q}^a x\neg\theta; \\ s_1 \models Q^a x\theta, & \quad s_2 \models \check{Q}^a x\neg\theta, \end{aligned}$$

a contradiction. This finishes the (sketchy) proof of the lemma.

PROOF OF THE THEOREM. By the Model Existence Theorem, $\{\phi, \neg\psi\} \notin S$. Hence there is a θ such that

- (1) $\phi \models \theta$, $\neg\psi \models \neg\theta$.
- (2) $F(\theta) \subset F(\phi) \cap F(\neg\psi)$ when F is Rel, Cn, Sort and Qu.
- (3) $\text{Un}(\theta) \subset \text{Un}(\phi)$, $\text{Ex}(\theta) \subset \text{Un}(\neg\psi)$, and similarly for Un^* and Ex^* .

This θ is the desired interpolant.

5. Preservation theorem. A major obstacle to theorems about extensions of $L(Q)$ structures has turned out to be the very definition of extensions; indeed we know of no wholly satisfactory definition as yet. For end extensions however, there is a good definition in Barwise [2].

DEFINITION. $(\mathfrak{M}, q) \subset_{\text{end}} (\mathfrak{N}, r)$ iff $\mathfrak{M} \subset_{\text{end}} \mathfrak{N}$ and for all $j \in J$ and $a \in M$, whenever $X \subset a_E$ then $X \in q(j, a)$ iff $X \in r(j, a)$.

DEFINITION. The Σ formulas are those ϕ such that $\text{Un}^*(\phi) = 0$.

LEMMA. *The Σ sentences persist for end extensions.*

PROOF. By induction on complexity.

We will show that conversely any persistent formula is equivalent to a Σ formula. To do this we first need an approximation to the definition of end extension which can be expressed in two-sorted language within $L_{\omega, \omega}(Q)$.

DEFINITION. Let L_A be a fragment of $L_{\omega, \omega}(Q)$. Then $(\mathfrak{M}, q) \subset_A (\mathfrak{N}, r)$ iff $\mathfrak{M} \subset_{\text{end}} \mathfrak{N}$ and for each $j \in J$ and $X \subset a_E$: If X is definable in (\mathfrak{M}, q) by a formula of L_A then $X \in q(j, a)$ implies $X \in r(j, a)$ and $X \in \check{q}(j, a)$ implies $X \in \check{r}(j, a)$.

LEMMA. *If L_A is a fragment of $L_{\omega, \omega}(Q)$ and $\phi \in L_A$ persists for end extensions, then ϕ persists for \subset_A extensions.*

PROOF. Let \underline{M} be a new unary relation symbol. For any formula ψ let ψ^M denote the relativization of ψ to \underline{M} , i.e., replacing ' $\forall x$ ' by ' $\forall x(\underline{M}(x) \rightarrow \dots)$ ' and ' $\exists x$ ' by ' $\exists x(\underline{M}(x) \wedge \dots)$ '. Given $(\mathfrak{M}, q) \subset_A (\mathfrak{N}, r)$ we will produce quantifiers \hat{q} and r^d such that $(\mathfrak{M}, q) \equiv_A (\mathfrak{M}, \hat{q}) \subset_{\text{end}} (\mathfrak{N}, r^d) \equiv_A (\mathfrak{N}, r)$.

Consider the $L(\underline{M})$ structure $\mathfrak{N} = (\mathfrak{N}, M, r)$ where M is the universe of the model \mathfrak{N} . We form a new structure $\mathfrak{N}^d = (\mathfrak{N}, M, r^d)$, where $r^d(j, b)$ is the monotone quantifier generated by the definable (in \mathfrak{N}) elements of $r(j, b)$. Then, as in [2], $\mathfrak{N} \equiv_A \mathfrak{N}^d$ and so $(\mathfrak{N}, r^d) \equiv_A (\mathfrak{N}, r)$. Let $\hat{q}(j, a)$ be the monotone quantifier generated by $\{X \subset a_E \mid X \in r^d(j, a)\}$; thus $(\mathfrak{M}, \hat{q}) \subset_{\text{end}} (\mathfrak{N}, r^d)$. It remains to show $(\mathfrak{M}, q) \equiv_A (\mathfrak{M}, \hat{q})$, for which it suffices to show that $q(j, a)$ and $\hat{q}(j, a)$ have the same definable elements. We will need the following two facts, both of which can be proved by induction on complexity. Let ψ be an L_A formula in one free variable with parameters from M . Then

- (i) $\{x \mid (\mathfrak{M}, q) \vDash \psi(x)\} = \{x \in M \mid \mathfrak{N} \vDash \psi^M(x)\}$;
- (ii) $\{x \mid (\mathfrak{M}, \hat{q}) \vDash \psi(x)\} = \{x \in M \mid \mathfrak{N} \vDash \psi^M(x)\}$.

Let $X \subset a_E$.

Case 1. Suppose $X \in q(j, a)$ and is definable in (\mathfrak{M}, q) , say $X = \{x \mid (\mathfrak{M}, q) \vDash \psi(x)\}$. Then $X \in r(j, a)$. Moreover, by (i), X is definable in \mathfrak{N} by the formula ψ^M , so $X \in r^d(j, a)$ and hence $X \in \hat{q}(j, a)$.

Case 2. Suppose $X \in \hat{q}(j, a)$ and $X = \{x \mid (\mathfrak{M}, \hat{q}) \vDash \psi(x)\}$. By (i) and (ii), $X = \{x \mid (\mathfrak{M}, q) \vDash \psi(x)\}$ and so X is definable in (\mathfrak{M}, q) . Suppose $X \notin q(j, a)$. Then $a_E - X \in \check{q}(j, a)$ and so $a_E - X \in \check{r}(j, a)$. Hence $X \notin r(j, a)$, $X \notin r^d(j, a)$ and so $X \notin \hat{q}(j, a)$, a contradiction.

This completes the proof that $(\mathfrak{M}, q) \equiv_A (\mathfrak{M}, \hat{q})$ and hence the proof of the lemma.

THEOREM. *Let L be a single-sorted language with no function symbols and suppose ϕ is a sentence of some countable admissible fragment L_A . If ϕ persists for end extensions then ϕ is equivalent to a Σ sentence of L_A .*

PROOF. Expand L to L^+ by introducing \aleph_0 new variables of a second sort and a new constant c' for each constant c of L . This should be done in a reasonable way; in particular, the new symbols should all be in the admissible set A . We use x, y, z, \dots to denote the original variables, x', y', z', \dots their counterparts of the second sort. For any formula ψ of L_A let ψ' denote the result of replacing each variable and constant by its primed counterpart. We take $J = \{0, 1\}$.

Let EXT be the collection of the following sentences of L_A^+ :

$$\forall x \exists x'(x = x');$$

$$\forall x \forall x' \in x \exists y(y = x');$$

$$(c = c'), \text{ for each constant } c \text{ of } L;$$

$$\forall x (Q^x y \psi(y) \leftrightarrow Q^x y' \psi(y')), \text{ for each } \psi \text{ of } L_A.$$

Any model of EXT will be of the form $(M, M', c_i, c'_i, R_i, q)$ and moreover $(M, c_i, R_i \upharpoonright M, q \upharpoonright \{0\} \times M) \subset_A (M', c'_i, R_i \upharpoonright M', q \upharpoonright \{1\} \times M)$. Thus if ϕ persists for end extensions and hence \subset_A extensions, we have $\text{EXT} \models \phi \rightarrow \phi'$. Now by the Barwise Compactness Theorem, there is a subset $\text{EXT}_0 \subset \text{EXT}$ such that EXT_0 is in A and $\models \bigwedge \text{EXT}_0 \wedge \phi \rightarrow \phi'$. By the interpolation theorem there is a θ' of L_A^+ such that

$$\models \bigwedge \text{EXT}_0 \wedge \phi \rightarrow \theta' \text{ and } \models \theta' \rightarrow \phi',$$

$$\text{Un}^*(\theta') \subset \text{Un}^*(\text{EXT}_0 + \phi) \subset \{0\},$$

$$\text{Sort}(\theta') \subset \text{Sort}(\phi') = \{1\}.$$

The last two imply that $\text{Un}^*(\theta) = 0$, while the first implies $\models \theta \leftrightarrow \phi$ as desired.

BIBLIOGRAPHY

1. Jon Barwise, *Admissible sets and structures. An approach to definability theory*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin-New York, 1975.
2. ———, *Monotone quantifiers and admissible sets*, Generalized Recursion Theory. II, North-Holland, Amsterdam, 1978.
3. Kim Bruce, *Ideal models and some not so ideal problems in the model theory of $L(Q)$* , J. Symbolic Logic 43 (1978), 304–321.
4. Solomon Feferman, *Applications of many-sorted interpolation theorems*, Proceedings of the Tarski Symposium, (Proc. Sympos. Pure Math., vol. 25, Univ. of California, Berkeley, 1971), Amer. Math. Soc., Providence, R.I., 1974, pp. 205–223.
5. H. J. Keisler, *Model theory for infinitary logic*, North-Holland, Amsterdam, 1971.
6. Bienvenido F. Nebres, *Infinitary formulas preserved under unions*, J. Symbolic Logic 37 (1972), 449.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, MADISON, WISCONSIN 53706