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DESINGULARIZING MAPS OF CORANK ONE

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ABSTRACT. In 1960, A. Haefliger presented necessary and sufficient conditions for factoring a smooth map from a surface into the plane through an immersion into R^3. Here, necessary and sufficient conditions are given for factoring a map between manifolds of dimension n, $n \geqslant 2$, through an immersion into a line bundle over the range. In addition, conditions are given for factoring such a map through a submersion from a line bundle over the domain.

By a desingularization of a smooth map $f: M \to V$ between manifolds we mean one of the following.

(1) An immersion $F: M \to E$ for some line bundle $\pi: E \to V$ such that $f = \pi \circ F$ or

(2) a submersion $F': E' \to V$ for some line bundle $\pi': E' \to M$ with zero section s such that $f = F' \circ s$.

We say f desingularizes in the first (resp. second) sense with respect to E (resp. E'). If f has a desingularization in the first sense, then the pull-back of the normal bundle to the image of F provides a line bundle with respect to which f has a desingularization in the second sense. The converse is not true. The map for Figure 1 desingularizes in the second sense but not the first. In this paper we present necessary and sufficient conditions for desingularizing a generic map with respect to a given bundle. In [1] A. Haefliger provides necessary and sufficient conditions for factoring a generic map from a closed surface into the plane through an immersion into R^3. Let $f: M \to V$ be a smooth map where M is a closed smooth n-manifold and V is a smooth n-manifold and let $\pi: E \to V$ be a line bundle. If there is an immersion $F: M \to E$ with $f = \pi \circ F$ we say f factors through an immersion into E. If f does factor then rank $df \geqslant n - 1$ everywhere, or equivalently, the corank of f does not exceed one. The singularities of a generic f, by which we mean 2-generic and finite-to-one [2], with rank $df \geqslant n - 1$ everywhere, are stratified by the manifolds

$$S_1 = \{x \in M | \text{rank } df_x = n - 1\}, \qquad S_{1,1} = \{x \in S_1 | \text{rank } d(f_{|S_1})_x = n - 2\}.$$

The map f being 2-generic ensures that S_1 and $S_{1,1}$ are submanifolds of M.

If we regard the Klein bottle as $\{(\phi, \theta) | \phi \in S^1, \theta \in [0, 2\pi]\}$ mod the relation $(\phi, 0) = (-\phi, 2\pi)$ then $f(\phi, \theta) = (\cos \phi + 2, \theta)$ is a map from the Klein bottle to the plane (in polar coordinates). We will show that this map desingularizes in the second sense but not in the first sense. Figure 1 represents the image of this map.

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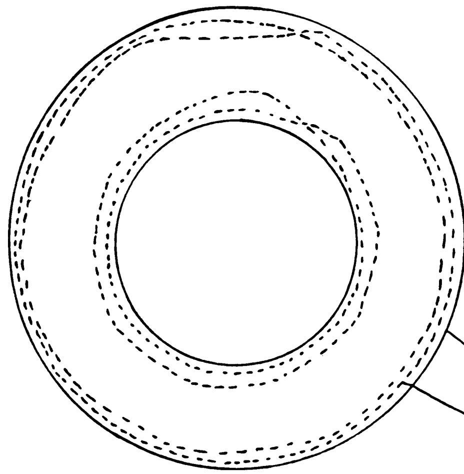

FIGURE 1: Image of f: (Klein bottle) $\longrightarrow \mathbf{R}^2$

For $n = 2$ a point in $S_{1,1}$ is called a cusp. In this setting Haefliger's theorem is: *A generic map f: $M^2 \to R^2$ factors through an immersion into R^3 if and only if each connected component of S_1 with an even (resp. odd) number of cusps has an orientable (resp. nonorientable) neighborhood.*

The following example of a map f: $S^2 \to R^2$ which cannot be desingularized is due to A. Haefliger:

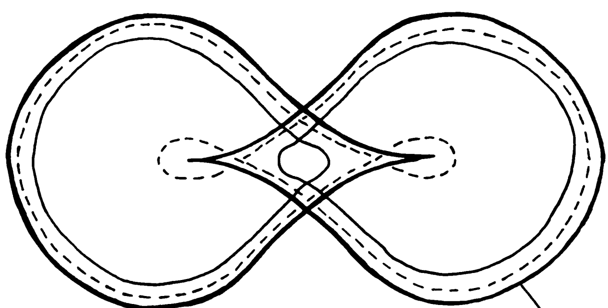

FIGURE 2

The set S_1 consists of two parallel curves C' and C'' each containing exactly one cusp; the image of the annulus bounded by two parallel curves between C' and C'' is a ribbon forming a figure eight bounded by the dashed curves; the two thin curves represent the images of the boundaries of polar caps.

For $n > 2$ there is the following theorem due to Y. Saito [3]: *A generic map f: $M^n \to R^n$ factors through an immersion into R^{n+1} if* (i) *M is orientable,* (ii) rank $df > n - 1$ *everywhere and* (iii) $S_{1,1} = \varnothing$.

Let $N \to S_1$ be a tubular neighborhood of S_1 and let $K \to S_1$ be the line bundle given by the kernel of df if rank $df \geqslant n-1$ everywhere. We denote the first Stiefel-Whitney classes of these bundles by $w(N)$ and $w(K)$ respectively. Similarly we denote the class of the bundle $E \to V$ by $w(E)$. Let $[S_{1,1}]$ be the class in $H^1(S_1; Z_2)$ dual to the homology class represented by $S_{1,1}$ and let i: $S_1 \to M$ be the inclusion map.

THEOREM 1. *A generic map f: $M \to V$ factors through an immersion into E if and only if the following conditions hold.*

(i) rank $df \geqslant n-1$ *everywhere and*

(ii) $[S_{1,1}] + w(N) + i^*f^*w(E) = 0$.

REMARK. If M is orientable and E is trivial, then condition (ii) is equivalent to the statement that $S_{1,1}$ bounds in S_1.

PROOF OF THEOREM 1. Even if f does not factor, we have $[S_{1,1}] + w(N) + w(K) = 0$.

We present an elementary proof of this fact. Let α: $K \to N$ be the restriction to K of the projection map from the tangent space onto the normal bundle and let σ: $S_1 \to K$ be a section transverse to the zero section of K. Since f is 2-generic the section $\alpha \circ \sigma$ is transverse to the zero section of N. The homology class dual to $w(K)$ is represented by the intersection of α and the zero section. Similarly we can represent the dual of $w(N)$ by the intersection of $\alpha \circ \sigma$ and the zero section in N. Since $\alpha \circ \sigma(x)$ is in the zero section if and only if either $x \in S_{1,1}$ or $\sigma(x)$ is in the zero section, these homology classes are equal. Hence $[S_{1,1}] + w(K) = w(N)$.

If f factors through an immersion F, then dF defines a bundle map from K to E over $f \circ i$, which is an isomorphism on the fibres. Hence $i^*f^*w(E) = w(K)$.

Conversely, if $i^*f^*w(E) = w(K)$ there is a bundle map L: $K \to E$ over $f \circ i$ which is an isomorphism on the fibres. We will use this map in factoring f locally. For each $p \in M$ choose a closed disk B_p containing p such that the bundles K and E are trivial over $B_p \cap S_1$ and $f(B_p)$ respectively. Furthermore, since f is finite-to-one we can choose these disks so that $f(p) \notin f(\partial B_p)$. From this first condition we obtain, by integrating a vector field extending K, an immersion F_p: $B_p \to E$ for each p. Furthermore we require these immersions to satisfy $f = \pi \circ F_p$ and $d(F_p)_x(k) = \lambda_{p,x} L(k)$ where $x \in S_1 \cap B_p$, $k \in K$ and $\lambda_{p,x} > 0$. Here we identify a vector v from the tangent space of E with a vector in E if $d\pi(v) = 0$. From the second condition and the compactness of M we can cover M with finitely many open disks $D_1, D_2, \ldots, D_r$ such that $\overline{D_j} \subset B_{p_j}$ and $f(\overline{D_j}) \cap f(\partial B_{p_j}) = \varnothing$ for each j and corresponding p_j.

Next choose smooth maps ϕ_j: $V \to R^1$ such that (i) $\phi_j \geqslant 0$, (ii) $\phi_j \circ f \equiv 0$ near ∂B_{p_j} and (iii) $\phi_j \circ f > 0$ on D_j. For each j let σ_j: $M \to R^1$ be the map defined by $\sigma_j(x) = \phi_j \circ f(x)$ for $x \in B_{p_j}$ and $\sigma_j(x) = 0$ otherwise. Let $\sigma_j F_{p_j}$: $M \to E$ be the fibre product of σ_j and F_{p_j} and let $F = \sum_{j=1}^{r} \sigma_j F_{p_j}$ be the fibre sum. Note that ker $dF \subset$ ker $d(\pi \circ F) =$ ker df. Since df is 1-1 away from S_1, so is dF. Since dim ker $df_x = 1$ for $x \in S_1$ it suffices to show $dF_x(k) \neq 0$ for k a nonzero vector in the kernel of df_x. We will calculate dF in a neighborhood of x in which each F_{p_j} is of the form (f, h_j) where h_j maps into the fibre of E. Then

$$dF_x(k) = \sum_{j=1}^{r} d(\sigma_j F_{p_j})_x(k) = \sum_{j=1}^{r} d(\sigma_j h_j)_x(k) = \sum_{j=1}^{r} \sigma_j d(h_j)_x(k)$$
$$= \sum_{j=1}^{r} \sigma_j \lambda_{p,x} L(k) = \mu L(k)$$

where $\mu > 0$. Therefore F is an immersion.

We now consider desingularization in the second sense. For a given line bundle $\pi'\colon E' \to M$ with zero section $s\colon M \to E'$ we have

THEOREM 2. *For a generic map $f\colon M \to V$ there is a submersion $F'\colon E' \to V$ with $f = F' \circ s$ if and only if the following conditions hold.*

(i) rank $df \geqslant n - 1$ *everywhere and*

(ii) $[S_{1,1}] = i^* w(E')$.

PROOF OF THEOREM 2. At a point $p \in S_1 - S_{1,1}$ the map f is a *fold*, i.e. f can be written as $f(x, y) = (x^2, y)$ for $x \in R^1$ and $y \in R^{n-1}$. If we have a submersion F' there is a unique *positive* direction on the fibre $E'_{p'}$, $p \in S_1 - S_{1,1}$ determined by: v is *positive* if and only if for any neighborhood of p, N_p, we have $F(\alpha v) \in f(N_p)$ for all sufficiently small $\alpha > 0$. Since f is 2-generic we can choose a section of $E'_{|S_1}$ which is positive on $S_1 - S_{1,1}$ and which meets the zero section transversally with intersection $S_{1,1}$. Hence $[S_{1,1}]$ is the Stiefel-Whitney class of $E'_{|S_1}$ as is $i^* w(E')$.

Conversely if $[S_{1,1}] = i^* w(E')$ consider the line bundle over S_1 whose fibres are the lines normal to the image $f(S_1)$. For a 2-generic f this latter bundle is well defined and has Stiefel-Whitney class $[S_{1,1}]$. Therefore this bundle is equivalent to $E'_{|S_1}$. Using a bundle isomorphism we can define F' on $E'_{|S_1}$. We then extend F' so that it is constant on the fibres outside of a neighborhood of S_1.

Since any class in $H^1(M; Z_2)$ is the Stiefel-Whitney class of some line bundle over M, the following are immediate for a map f with rank $df > n - 1$ everywhere.

COROLLARY 1. *A generic $f\colon M \to V$ desingularizes in the first sense if and only if there is a class $w \in H^1(V; Z_2)$ which satisfies the equation $i^* f^*(w) + [S_{1,1}] + w(N) = 0$.*

COROLLARY 2. *A generic $f\colon M \to V$ desingularizes in the second sense if and only if there is a class $w \in H^1(M; Z_2)$ which satisfies the equation $i^*(w) = [S_{1,1}]$.*

With these corollaries we can establish the desingularization statements for the map of Figure 1.

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