DESINGULARIZING MAPS OF CORANK ONE

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Abstract. In 1960, A. Haefliger presented necessary and sufficient conditions for factoring a smooth map from a surface into the plane through an immersion into $R^3$. Here, necessary and sufficient conditions are given for factoring a map between manifolds of dimension $n, n > 2$, through an immersion into a line bundle over the range. In addition, conditions are given for factoring such a map through a submersion from a line bundle over the domain.

By a desingularization of a smooth map $f: M \to V$ between manifolds we mean one of the following.

1. An immersion $F: M \to E$ for some line bundle $\pi: E \to V$ such that $f = \pi \circ F$
2. A submersion $F': E' \to V$ for some line bundle $\pi': E' \to M$ with zero section $s$ such that $f = F' \circ s$.

We say $f$ desingularizes in the first (resp. second) sense with respect to $E$ (resp. $E'$). If $f$ has a desingularization in the first sense, then the pull-back of the normal bundle to the image of $F$ provides a line bundle with respect to which $f$ has a desingularization in the second sense. The converse is not true. The map for Figure 1 desingularizes in the second sense but not the first. In this paper we present necessary and sufficient conditions for desingularizing a generic map with respect to a given bundle. In [1], A. Haefliger provides necessary and sufficient conditions for factoring a generic map from a closed surface into the plane through an immersion into $R^3$. Let $f: M \to V$ be a smooth map where $M$ is a closed smooth $n$-manifold and $V$ is a smooth $n$-manifold and let $\pi: E \to V$ be a line bundle. If there is an immersion $F: M \to E$ with $f = \pi \circ F$ we say $f$ factors through an immersion into $E$. If $f$ does factor then rank $df > n - 1$ everywhere, or equivalently, the corank of $f$ does not exceed one. The singularities of a generic $f$, by which we mean 2-generic and finite-to-one [2], with rank $df > n - 1$ everywhere, are stratified by the manifolds

$S_1 = \{x \in M | \text{rank } df_x = n - 1\}, \quad S_{1,1} = \{x \in S_1 | \text{rank } d(f|_{S_1})_x = n - 2\}$

The map $f$ being 2-generic ensures that $S_1$ and $S_{1,1}$ are submanifolds of $M$.

If we regard the Klein bottle as $\{((\phi, \theta))|\phi \in S^1, \theta \in [0, 2\pi]\}$ mod the relation $(\phi, 0) = (-\phi, 2\pi)$ then $f(\phi, \theta) = (\cos \phi + 2, \theta)$ is a map from the Klein bottle to the plane (in polar coordinates). We will show that this map desingularizes in the second sense but not in the first sense. Figure 1 represents the image of this map.
For $n = 2$ a point in $S_{1,1}$ is called a cusp. In this setting Haefliger's theorem is: A generic map $f: M^2 \to R^2$ factors through an immersion into $R^3$ if and only if each connected component of $S_1$ with an even (resp. odd) number of cusps has an orientable (resp. nonorientable) neighborhood.

The following example of a map $f: S^2 \to R^2$ which cannot be desingularized is due to A. Haefliger:

For $n > 2$ there is the following theorem due to Y. Saito [3]: A generic map $f: M^n \to R^n$ factors through an immersion into $R^{n+1}$ if (i) $M$ is orientable, (ii) rank $df > n - 1$ everywhere and (iii) $S_{1,1} = \emptyset$.

Let $N \to S_1$ be a tubular neighborhood of $S_1$ and let $K \to S_1$ be the line bundle.
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given by the kernel of \( df \) if \( \text{rank } df > n - 1 \) everywhere. We denote the first Stiefel-Whitney classes of these bundles by \( w(N) \) and \( w(K) \) respectively. Similarly we denote the class of the bundle \( E \rightarrow V \) by \( w(E) \). Let \([S_{1,1}]\) be the class in \( H^1(S_1; \mathbb{Z}_2) \) dual to the homology class represented by \( S_{1,1} \) and let \( i: S_1 \rightarrow M \) be the inclusion map.

**Theorem 1.** A generic map \( f: M \rightarrow V \) factors through an immersion into \( E \) if and only if the following conditions hold.

(i) \( \text{rank } df > n - 1 \) everywhere and
(ii) \([S_{1,1}] + w(N) + i^* f^* w(E) = 0.\)

**Remark.** If \( M \) is orientable and \( E \) is trivial, then condition (ii) is equivalent to the statement that \( S_{1,1} \) bounds in \( S_1 \).

**Proof of Theorem 1.** Even if \( f \) does not factor, we have \([S_{1,1}] + w(N) + w(K) = 0.\)

We present an elementary proof of this fact. Let \( \alpha: K \rightarrow N \) be the restriction to \( K \) of the projection map from the tangent space onto the normal bundle and let \( \sigma: S_1 \rightarrow K \) be a section transverse to the zero section of \( K \). Since \( f \) is 2-generic the section \( \alpha \circ \sigma \) is transverse to the zero section of \( N \). The homology class dual to \( w(K) \) is represented by the intersection of \( \alpha \) and the zero section. Similarly we can represent the dual of \( w(N) \) by the intersection of \( \alpha \circ \sigma \) and the zero section in \( N \).

Since \( \alpha \circ \sigma(x) \) is in the zero section if and only if either \( x \in S_{1,1} \) or \( \sigma(x) \) is in the zero section, these homology classes are equal. Hence \([S_{1,1}] + w(K) = w(N).\)

If \( f \) factors through an immersion \( F \), then \( dF \) defines a bundle map from \( K \) to \( E \) over \( f \circ i \), which is an isomorphism on the fibres. Hence \( i^* f^* w(E) = w(K). \)

Conversely, if \( i^* f^* w(E) = w(K) \) there is a bundle map \( L: K \rightarrow E \) over \( f \circ i \) which is an isomorphism on the fibres. We will use this map in factoring \( f \) locally.

For each \( p \in M \) choose a closed disk \( B_p \) containing \( p \) such that the bundles \( K \) and \( E \) are trivial over \( B_p \cap S_1 \) and \( f(B_p) \) respectively. Furthermore, since \( f \) is finite-to-one we can choose these disks so that \( f(p) \notin \mathcal{E}(\partial B_p) \). From this first condition we obtain, by integrating a vector field extending \( K \), an immersion \( F_p: B_p \rightarrow E \) for each \( p \). Furthermore we require these immersions to satisfy \( f = \pi \circ F_p \) and \( d(F_p)_x(k) = \lambda_{p,x} L(k) \) where \( x \in S_1 \cap B_p \), \( k \in K \) and \( \lambda_{p,x} > 0 \). Here we identify a vector \( v \) from the tangent space of \( E \) with a vector in \( \mathcal{E} \) if \( d\pi(v) = 0 \). From the second condition and the compactness of \( M \) we can cover \( M \) with finitely many open disks \( D_1, D_2, \ldots, D_r \) such that \( \overline{D_j} \subset B_{p_j} \) and \( f(\partial D_j) \cap f(\partial B_{p_j}) = \emptyset \) for each \( j \) and corresponding \( p_j \).

Next choose smooth maps \( \phi_j: V \rightarrow R^1 \) such that (i) \( \phi_j > 0 \), (ii) \( \phi_j \circ f \equiv 0 \) near \( \partial B_{p_j} \) and (iii) \( \phi_j \circ f > 0 \) on \( D_j \). For each \( j \) let \( \sigma_j: M \rightarrow R^1 \) be the map defined by \( \sigma_j(x) = \phi_j \circ f(x) \) for \( x \in B_{p_j} \) and \( \sigma_j(x) = 0 \) otherwise. Let \( \sigma_j F_j: M \rightarrow E \) be the fibre product of \( \sigma_j \) and \( F_p \) and let \( F = \bigcup_{j=1}^n \sigma_j F_j \) be the fibre sum. Note that \( \ker dF \subset \ker d(\pi \circ F) = \ker df \). Since \( df \) is 1-1 away from \( S_1 \), so is \( dF \). Since \( \dim \ker dF_x = 1 \) for \( x \in S_1 \) it suffices to show \( dF_x(k) \neq 0 \) for \( k \) a nonzero vector in the kernel of \( dF_x \). We will calculate \( dF \) in a neighborhood of \( x \) in which each \( F_{p_j} \) is of the form \( (f, h_j) \) where \( h_j \) maps into the fibre of \( E \). Then
\[ dF_x(k) = \sum_{j=1}^{r} d(\sigma_j F_p)_x(k) = \sum_{j=1}^{r} d(\sigma_j h_j)_x(k) = \sum_{j=1}^{r} \sigma_j d(h_j)_x(k) \]

\[ = \sum_{j=1}^{r} \sigma_j \lambda_{p,x}(k) \mu L(k) \]

where \( \mu > 0 \). Therefore \( F \) is an immersion.

We now consider desingularization in the second sense. For a given line bundle \( \pi': E' \to M \) with zero section \( s: M \to E' \) we have

**THEOREM 2.** For a generic map \( f: M \to V \) there is a submersion \( F': E' \to V \) with \( f = F' \circ s \) if and only if the following conditions hold.

(i) rank \( df \geq n - 1 \) everywhere and

(ii) \([S_{1,1}] = i^*w(E')\).

**Proof of Theorem 2.** At a point \( p \in S_1 - S_{1,1} \) the map \( f \) is a fold, i.e. \( f \) can be written as \( f(x, y) = (x^2, y) \) for \( x \in \mathbb{R}^1 \) and \( y \in \mathbb{R}^{n-1} \). If we have a submersion \( F' \) there is a unique positive direction on the fibre \( E'_p, p \in S_1 - S_{1,1} \) determined by: \( v \) is positive if and only if for any neighborhood of \( p, N_p \), we have \( F(av) \in f(N_p) \) for all sufficiently small \( \alpha > 0 \). Since \( f \) is 2-generic we can choose a section of \( E'|_S \), which is positive on \( S_1 - S_{1,1} \) and which meets the zero section transversally with intersection \( S_{1,1} \). Hence \([S_{1,1}] \) is the Stiefel-Whitney class of \( E'|_S \) as is \( i^*w(E') \).

Conversely if \([S_{1,1}] = i^*w(E')\) consider the line bundle over \( S_1 \) whose fibres are the lines normal to the image \( f(S_1) \). For a 2-generic \( f \) this latter bundle is well defined and has Stiefel-Whitney class \([S_{1,1}] \). Therefore this bundle is equivalent to \( E'|_S \). Using a bundle isomorphism we can define \( F' \) on \( E'|_S \). We then extend \( F' \) so that it is constant on the fibres outside of a neighborhood of \( S_1 \).

Since any class in \( H^1(M; \mathbb{Z}_2) \) is the Stiefel-Whitney class of some line bundle over \( M \), the following are immediate for a map \( f \) with rank \( df \geq n - 1 \) everywhere.

**COROLLARY 1.** A generic \( f: M \to V \) desingularizes in the first sense if and only if there is a class \( w \in H^1(V; \mathbb{Z}_2) \) which satisfies the equation \( i^*f^*(w) + [S_{1,1}] + w(N) = 0 \).

**COROLLARY 2.** A generic \( f: M \to V \) desingularizes in the second sense if and only if there is a class \( w \in H^1(M; \mathbb{Z}_2) \) which satisfies the equation \( i^*(w) = [S_{1,1}] \).

With these corollaries we can establish the desingularization statements for the map of Figure 1.

**REFERENCES**