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A CHARACTERIZATION OF COABSOLUTENESS FOR A CLASS OF METRIC SPACES

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ABSTRACT. All regular Hausdorff topological spaces can be partitioned into classes of coabsolute spaces. It is shown that in studying the coabsolute classes, only two types of spaces need to be considered: (i) those spaces which have a dense subset of locally compact points and (ii) nowhere locally compact spaces. The absolute of a space is the disjoint union of the absolute of a space of type (i) and the absolute of a space of type (ii). A characterization of the coabsolute subclasses is given for a class of metric spaces of type (i).

It is assumed that all spaces are regular Hausdorff. For each regular Hausdorff space X, there is a unique (up to a homeomorphism) extremally disconnected space EX called the absolute of X that can be mapped onto X by a closed irreducible, perfect, continuous map. The modern theory of the absolute was developed primarily by A. M. Gleason [5] and S. Iliadis [6]. Since the composition of closed irreducible perfect maps is closed irreducible perfect, and since the absolute of a space is unique (up to a homeomorphism), (1) if there is a closed irreducible perfect continuous map from X onto Y then EX and EY are homeomorphic. Two spaces for which the absolutes are homeomorphic are said to be *coabsolute spaces*. Obviously all regular topological spaces can be partitoned into classes of coabsolute spaces. Recently, properties common to spaces within a coabsolute class and conditions necessary and/or sufficient to make spaces coabsolute have been under study. This author became interested in characterizing coabsoluteness of spaces upon determining that all normal spaces within a coabsolute class have homeomorphic sets of remote points [4].

In the first section it will be seen that in studying the coabsolute classes, only two types of spaces need to be considered: (i) those spaces which have a dense subset of locally compact points and (ii) nowhere locally compact spaces. It is shown in Theorem 1.4 that the absolute of a space is the disjoint union of the absolute of a space of type (i) and the absolute of a space of type (ii). In the second section a characterization of the coabsolute subclasses is given for a class of metric spaces of type (i).

1. Decomposition of EX. Let X be a regular Hausdorff space. The *locally compact points* of X are all those points $x \in X$ for which there exists an open neighborhood U in X with $\mathrm{cl}_X U$ compact. A space is *nowhere locally compact* if no point has a

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compact neighborhood. LX will denote the locally compact points of X; NX will denote $X \setminus \mathrm{cl}_X LX$. Clearly LX and NX have the following properties: LX and NX are disjoint open subsets of X and NX is regular open, $\mathrm{cl}_X LX$ and $\mathrm{cl}_X NX$ are regular closed subsets of X, $X = \mathrm{cl}_X LX \cup \mathrm{cl}_X NX$, $\mathrm{cl}_X LX \cap \mathrm{cl}_X NX$ is a closed nowhere dense subset of X and thus $\mathrm{cl}_X LX$ and $\mathrm{cl}_X NX$ have disjoint interiors. Some other properties of $\mathrm{cl}_X LX$ and $\mathrm{cl}_X NX$ are given in Lemma 1.1.

LEMMA 1.1. *Let X be a space.* (i) *If $V \subseteq X$ is open and $V \subseteq NX$, then $\mathrm{cl}_X V$ is a nowhere locally compact space. In particular, $\mathrm{cl}_X NX$ is a nowhere locally compact space.*

(ii) *The locally compact points of $\mathrm{cl}_X LX$ are dense in $\mathrm{cl}_X LX$.*

DEFINITION 1.2. A continuous surjection f from a space X onto a space Y is *closed irreducible* if the image under f of every proper closed subset of X is a proper closed subset of Y.

If U and V are topological spaces, $U + V$ will denote the disjoint union of U and V.

LEMMA 1.3. *Let X be a space and let j: $\mathrm{cl}_X NX + \mathrm{cl}_X LX \to X$ be the map with $j|\mathrm{cl}_X NX$ and $j|\mathrm{cl}_X LX$ the identity maps on $\mathrm{cl}_X NX$ and $\mathrm{cl}_X LX$ respectively. Then j is a closed irreducible perfect continuous surjection.*

The decomposition mentioned in the introduction is given in the following theorem.

THEOREM 1.4. *Let X be a space. Then EX is homeomorphic to $EY + EW$ where Y is a space for which the set of locally compact points is dense and W is a nowhere locally compact space.*

PROOF. Let $Y = \mathrm{cl}_X LX$ and $W = \mathrm{cl}_X NX$. By Lemma 1.1, Y and W have the local compactness properties required in the theorem. By Lemma 1.3 there is a closed irreducible perfect continuous map from $Y + W$ onto X. By (1) in the introduction, this implies that X and $Y + W$ are coabsolute, so EX is homeomorphic to $E(Y + W)$. Let π_Y: $EY \to Y$ and π_W: $EW \to W$ be closed irreducible perfect continuous maps. It is easy to see that π: $EY + EW \to Y + W$ defined by $\pi(t) = \pi_Y(t)$ if $t \in Y$ and $\pi(t) = \pi_W(t)$ if $t \in W$ is also closed irreducible, perfect and continuous. Since $EY + EW$ is extremally disconnected and π: $EY + EW \to Y + W$ is a closed irreducible perfect continuous map, $EY + EW$ is homeomorphic to the absolute of $Y + W$ because of the uniqueness of the absolute. Thus EX and $E(Y + W)$ are homeomorphic.

2. Characterization of coabsoluteness. In this section a characterization of the coabsolute classes for a special class of metric spaces will be given. Lemma 2.2 contains some properties "preserved" by coabsolute classes, so each of these properties is shared by every member of the coabsolute class if any member has the property. If X is a space, dX denotes the density of X, i.e. the smallest cardinality of a dense subset of X.

LEMMA 2.1 ([**8**, 10.49]). *Let $f\colon X \to Y$ be a closed irreducible continuous map. If U is an open subset of X, then $Y \setminus f(X \setminus U) \subseteq f(U) \subseteq \mathrm{cl}_Y(Y \setminus f(X \setminus U))$, so $\mathrm{cl}_Y f(U) = \mathrm{cl}_Y(Y \setminus f(X \setminus U)) = f(\mathrm{cl}_X U)$.*

LEMMA 2.2. *Let X and Y be coabsolute spaces. Then*

(i) *$dX = dY$.*

(ii) ([**1**, XI 6.6]) *X is a locally compact if and only if Y is locally compact.*

(iii) *LX is a proper dense subset of X if and only if LY is a proper dense subset of Y.*

(iv) *$X \setminus LX$ is nonempty and compact if and only if $Y \setminus LY$ is nonempty and compact.*

PROOF. (i) If $f\colon Z \to X$ is a continuous map onto X and $D \subseteq Z$ is dense then $f(D)$ is dense in X and since the cardinality of $f(D)$ is less than or equal to that of D, $dX \leq dZ$. Suppose f is also closed irreducible and $E \subseteq X$ is dense. Let F be any set in Z such that $f(F) = E$ which has the same cardinality as E. Then since f is closed, $f(\mathrm{cl}_Z F) = \mathrm{cl}_X E = X$ and since f is irreducible, $\mathrm{cl}_Z F = Z$. So $dZ \leq dX$. Thus if there is a closed irreducible continuous map between two spaces, the spaces have the same density, so $dX = dEX$.

Let $f\colon W \subseteq X$ be a closed irreducible perfect continuous map. To prove (iii) and (iv) it suffices to prove the properties for W and X. Since LW is open in W and f is closed irreducible and continuous, $\mathrm{cl}_X(f(LW)) = \mathrm{cl}_X(X \setminus f(W \setminus LW))$ by Lemma 2.1. Furthermore, $X \setminus f(W \setminus LW)$ is locally compact. To see this let $x \in X \setminus f(W \setminus LW)$. Since $f^{-1}(x)$ is compact and $f^{-1}(x) \subseteq LW$, there is an open set U in W with $f^{-1}(x) \subseteq U \subseteq \mathrm{cl}_W U \subseteq LW$ for which $\mathrm{cl}_W U$ is compact. Thus $x \in X \setminus f(W \setminus U) \subseteq f(\mathrm{cl}_W U)$ which is compact. So x has a compact neighborhood and (1) $X \setminus f(W \setminus LW) \subseteq LX$. If LW is dense in W, then $X \setminus f(W \setminus LW)$ is dense in X (Lemma 2.1) and so by the preceding containment (1), LX is dense in X. If $W \setminus LW$ is compact, then $f(W \setminus LW)$ is compact and since $X \setminus LX \subseteq f(W \setminus LW)$ by (1) and $X \setminus LX$ is closed, $X \setminus LX$ is compact. On the other hand, it is easy to see that $f^{-1}(LX)$ is open and locally compact in W so (2) $f^{-1}(LX) \subseteq LW$. If LX is dense in X, then $f^{-1}(LX)$ is dense in W and therefore so is LW by (2). If $X \setminus LX$ is compact, then so is $f^{-1}(X \setminus LX)$ and since $W \setminus LW \subseteq W \setminus f^{-1}(LX) = f^{-1}(X \setminus LX)$ and $W \setminus LW$ is closed in W, $W \setminus LW$ is compact. The requirements in (iii) and (iv) involving proper sets follow easily from (ii).

The following characterization for coabsoluteness of locally compact metric spaces without isolated points is known (e.g. [**2**]).

THEOREM 2.3. *If X and Y are locally compact metric spaces without isolated points, then X and Y are coabsolute if and only if $dX = dY$.*

A similar characterization will be given for a class of metric spaces which are almost locally compact—in particular, the class of metric spaces without isolated points for which the set of points which are not locally compact points is compact and nowhere dense. If X and Y are two such spaces, the condition that $dX = dY$ is not strong enough to guarantee that the spaces be coabsolute. It must also be required that "small" open sets containing $X \setminus LX$ and $Y \setminus LY$ have the same density. The following definition makes this second requirement precise.

DEFINITION 2.4. Let C be a compact subset of a metric space X. Define $d_X(C) = \min\{dU: C \subseteq U,\ U \text{ open}\}$. Note that this is well defined since the cardinals are well ordered. Note further that for any open set U for which $C \subseteq U$, there exists an open set W with $C \subseteq W \subseteq U$ and $d_X C = dW$.

The characterization is given in Theorem 2.7. Some lemmas will be needed in the proof of Theorem 2.7.

Let $D(d)$ be the disjoint union of d copies of the Cantor space and let $\{D_n(d)\}$ be a countable set of copies of $D(d)$. Let $Z(d) = \Sigma_n D_n(d) \cup \{q\}$ where q is not a point in $\Sigma_n D_n(d)$, and define a topology on $Z(d)$ as follows. A set is open in $Z(d)$ if it is an open subset of $\Sigma_n D_n(d)$ or it it is the complement in $Z(d)$ of a closed set C in $\Sigma_n D_n(d)$ for which $C \subseteq \Sigma_{n=1}^N D_n(d)$ for some N. With this topology, $Z(d)$ is a regular Hausdorff space.

LEMMA 2.5. *Let X be a metric space without isolated points with $X \setminus LX = \{p\}$ and $d = dX = d_X\{p\}$. Then X is the closed irreducible perfect continuous image of $Z(d)$. Thus X and $Z(d)$ are coabsolute spaces.*

PROOF. Let $\{R_n\}$ be a countable collection of regular closed subsets of X whose interiors relative to X form a neighborhood base for p with $R_1 = X$, $R_n \supseteq \operatorname{int}_X R_n \supseteq R_{n+1}$, $R_n \cap X \setminus \operatorname{int}_X R_{n+1}$ not compact and $d(R_n \cap X \setminus \operatorname{int}_X R_{n+1}) = d$ for every n. Let $X_n = R_n \cap X \setminus \operatorname{int}_X R_{n+1}$. Since $X_n = \operatorname{cl}_X(\operatorname{int}_X R_n \cap X \setminus R_{n+1})$, X_n is a collection of regular closed subsets of X with pairwise disjoint interiors. For each n, $X_n \subseteq LX$ so X_n is a locally compact noncompact metric space without isolated points.

There exists a closed irreducible perfect continuous map from $D(d)$ onto any locally compact, noncompact metric space without isolated points which has density d ([7, Lemma 2.4]). Let $\Pi_n: D_n(d) \to X_n$ be a closed irreducible perfect continuous map for each n; then $\Sigma\, \Pi_n: \Sigma\, D_n(d) \to \Sigma\, X_n$ is closed irreducible, perfect and continuous. Let $\Pi: \Sigma\, X_n \to LX$ be defined by $\Pi(x) = x$ for $x \in \Sigma\, X_n$; since X_n is a neighborhood finite cover of LX with pairwise disjoint interiors, Π is closed irreducible, perfect and continuous. Finally, define $f: Z(d) \to X$ by $f|\Sigma\, D_n(d) = \Pi \circ \Sigma\, \Pi_n$ and $f(q) = p$. It is straightforward to show that f is continuous, closed irreducible and perfect.

It will be useful to keep the following facts about densities in mind. The proofs of these are straightforward.

LEMMA 2.6. (i) *$d(U) = d(C) = d(\operatorname{cl}_X U)$ where U is open in X and $U \subseteq C \subseteq \operatorname{cl}_X U$.*

(ii) *If $f: X \to Y$ is a continuous map, then $d(C) > d(f(C))$ for $C \subseteq X$.*

(iii) *If $f: X \to Y$ is a closed irreducible continuous map, the $d(U) = d(f^{-1}(U))$ for all U open in Y.*

THEOREM 2.7. *Let X and Y be metric spaces without isolated points for which $X \setminus LX$ and $Y \setminus LY$ are both nonempty, nowhere dense and compact. Then X and Y are coabsolute if and only if $dX = dY$ and $d_X(X \setminus LX) = d_Y(Y \setminus LY)$.*

PROOF. Suppose X and Y are coabsolute and let Z be a space with $f\colon Z \to X$, $g\colon Z \to Y$ closed irreducible continuous perfect maps. By Lemma 2.2 (i) $dX = dY$. Let U be an open subset of X with $X \setminus LX \subseteq U$ and $dU = d_X(X \setminus LX)$. For each $x \in LX$, let U_x be a separable neighborhood of x. Since $X = \bigcup_{x \in LX} U_x \cup U$, $Z = f^{-1}(X) = \bigcup_{x \in LX} f^{-1}(U_x) \cup f^{-1}(U)$. Now $g^{-1}(Y \setminus LY)$ is compact, so $g^{-1}(Y \setminus LY) \subseteq f^{-1}(U_{x_1}) \cup \cdots \cup f^{-1}(U_{x_n}) \cup f^{-1}(U)$ for some finite collection $\{x_1, \ldots, x_n\} \subseteq LX$. Let $W = U_{x_1} \cup \cdots \cup U_{x_n} \cup U$; so $g^{-1}(Y \setminus LY) \subseteq f^{-1}(W)$. Now $Y \setminus LY \subseteq Y \setminus g(Z \setminus f^{-1}(W)) \subseteq g(f^{-1}(W)) \subseteq \operatorname{cl}_Y(Y \setminus g(Z \setminus f^{-1}(W)))$ (Lemma 2.1) and $Y \setminus g(Z \setminus f^{-1}(W))$ is open in Y; so

$$\begin{aligned} d(W) &= d(f^{-1}(W)) \geqslant d(g(f^{-1}(W))) \\ &= d(Y \setminus g(Z \setminus f^{-1}(W))) \geqslant d_Y(Y \setminus LY) \end{aligned}$$

by Lemma 2.6. Furthermore, $d(W) = d(U)$ since $d(U_{x_i}) = \aleph_0$ for $i = 1, \ldots, n$; so $d_X(X \setminus LX) \geqslant d_Y(Y \setminus LY)$. A similar argument shows $d_Y(Y \setminus LY) \geqslant d_X(X \setminus LX)$. So the two conditions are necessary.

Let X be a metric space without isolated points with $X \setminus LX$ nonempty, nowhere dense, and compact. To show the sufficiency of the density conditions, it will be shown that X is coabsolute with a space that depends only on dX and $d_X(X \setminus LX)$.

In [3, Lemma 5.5] it is shown that X is coabsolute with a metric space Y for which $Y \setminus LY$ is a singleton; so without loss of generality it may be assumed that $X = LX \cup \{p\}$. Suppose that $dX = d_X(X \setminus LX)$. Then by Lemma 2.5, X is coabsolute with $Z(dX)$. If $dX > d_X(X \setminus LX)$, let R be a regular closed subset of X with $dR = d_X(X \setminus LX)$ and $X \setminus LX \subseteq \operatorname{int}_X R$. Then X is coabsolute with $R + X \setminus \operatorname{int}_X R$. Now $R \setminus LR = \{p\}$ and $dR = d_R\{p\} = d_X\{p\} = d_X(X \setminus LX)$, so by Lemma 2.5, ER depends only on $d_X(X \setminus LX)$. Also $X \setminus \operatorname{int}_X R$ is a locally compact, noncompact metric space without isolated points which has density dX. So by Theorem 2.3 $E(X \setminus \operatorname{int}_X R)$ depends only on dX. Since EX, $E(R + X \setminus \operatorname{int}_X R)$ and $ER + E(X \setminus \operatorname{int}_X R)$ are all homeomorphic spaces, EX depends only on dX and $d(X \setminus LX)$.

Note. The proof of the sufficiency of Theorem 2.7 which is presented in this paper is somewhat shorter than the author's original proof. The author wishes to acknowledge Eric van Douwen for his suggestion which resulted in this shortened proof.

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