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A NOTE ON THE EXISTENCE OF TORSION-FREE MODULES WITH A PRESCRIBED TYPE-SET

LUCIE DE MUNTER-KUYL

ABSTRACT. In [2], Ito has proved the existence of a completely anisotropic torsion-free abelian group of rank two with a prescribed type-set. We outline here an alternative construction in a more general setting.

The vocabulary and notations will be that of [1]. In addition, we refer the reader to [2] for the definition of relatively disjoint sets of superdivisors (i.e. characteristics) and types.

THEOREM 1. *Let A be a principal domain and let T be a relatively disjoint set of types. There exists a rank two torsion-free A-module M such that $T(M) = T$.*

PROOF. Let $\mathbb{S}$ be a relatively disjoint set of superdivisors representing T, let μ_0 be the GCD of any two distinct elements of $\mathbb{S}$, and let $\tau_0 = \tau(\mu_0)$. The only interesting case arises when card $\mathbb{S} > 3$.

Let $\mu_1, \mu_2 \in \mathbb{S}$, with μ_0, μ_1, μ_2 all distinct and set $\mathbb{S}^* = \mathbb{S} - \{\mu_0, \mu_1, \mu_2\}$. Let P be a complete set of pairwise nonassociated irreducible elements of A. For every $\mu \in \mathbb{S}^*$, let $P_\mu = \{p \in P;\ \mu(p) > \mu_0(p)\}$ and $P_\mu^\infty = \{p \in P_\mu;\ \mu(p) = \infty\}$. Let $P^* = \bigcup_{\mu \in \mathbb{S}^*} P_\mu$. There exists an injective mapping ψ: $\mu \mapsto k_\mu$ from $\mathbb{S}^* \cup \{\mu_0\}$ to $K^* = K - \{0\}$, such that $v_p(k_\mu) = 0$ for all $p \in P_\mu$. For each $p \in P^*$, choose $\rho(p) \in K_p$ satisfying $\rho(p) = k_\mu$ when $p \in P_\mu^\infty$, and $\rho(p) \in K_p - K$, with $v_p(\rho(p)) = 0$ and $v_p(\rho(p) - k_\mu) = \mu(p) - \mu_0(p)$, when $p \in P_\mu - P_\mu^\infty$. Then, select a sequence $(a_n(p))$ of elements of $A - (p)$, converging to $\rho(p)$, and such that $v_p(\rho(p) - a_n(p)) = n + \mu_0(p)$. Now, let x_1, x_2 be two independent elements of the A-module K^2 and let M be the A-submodule generated by the set $\{p^{-r}x_1, p^{-s}x_2$, for all $p \in P$ and $r, s \in \mathbf{N}, r < \mu_1(p), s < \mu_2(p)\} \cup \{p^{-(n+\mu_0(p))}(x_1 + a_n(p)x_2)$, for all $p \in P^*$ and $n \in \mathbf{N}\}$. Using Corollary 3.3 in [1], it is then readily checked that the module M has T for type-set.

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REMARK. If ψ is bijective, then $K(x_1 + k_{\mu_0}x_2) \cap M$ is the only rank one pure submodule of M of type τ_0, and M is thus completely anisotropic. Furthermore, if we suppose that μ_0 is not in $\mathbb{S}$, then a similar bijection from $\mathbb{S}^*$ onto K^* will give rise to a completely anisotropic module in which the type τ_0 is no more represented. In view of this, we need only build a bijection ψ: $\mu \mapsto k_\mu$ such that $v_p(k_\mu) = 0$ for all $p \in P_\mu$ to prove the following

THEOREM 2. *Let T be a relatively disjoint set of types and suppose* card T = card P = card K. *Then, there exists a completely anisotropic rank two torsion-free A-module M such that $T(M) = T$.*

Consider a well-ordering of P such that $P = \{p(\alpha)\}_{\alpha<\lambda}$, where λ = card P = card $\mathbb{S}^*$ = card K^* is a limit ordinal. Set $p_\mu = \inf P_\mu$. The well-ordering of P induces one on $\{p_\mu\}_{\mu\in\mathbb{S}^*}$, and therefore on $\mathbb{S}^*$. Let $\mathbb{S}^* = \{\mu(\alpha)\}_{\alpha<\lambda}$ and $\{p_\mu\}_{\mu\in\mathbb{S}^*} = \{p_{\mu(\alpha)}\}_{\alpha<\lambda}$.

Let ξ = card U, where U is the group of units of A, and well-order U so that $U = \{u(\beta)\}_{\beta<\nu}$, where $\nu = \xi + 1$, if ξ is finite and $\nu = \xi$, if ξ is a limit ordinal. Now, well-order K^* so that (i) $K^* = \{k(\alpha)\}_{\alpha<\lambda}$; (ii) for any pair of ordinals α and γ such that $\gamma \leq \alpha < \lambda$, $p(\alpha)$ does not divide $k(\gamma)$; (iii) for any limit ordinal $\alpha \leq \lambda$, and any choice of $\alpha_0 < \inf(\alpha, \nu)$, $n \in \mathbf{N}^*$, $r_1, \ldots, r_n \in \mathbf{Z}$ and $\alpha_1, \ldots, \alpha_n < \alpha$, there exists an ordinal $\beta < \alpha$ such that $k(\beta) = u(\alpha_0)p(\alpha_1)^{r_1} \ldots p(\alpha_n)^{r_n}$. The ensuing bijection ψ: $\mu(\alpha) \mapsto k(\alpha)$ has the desired property, proving the theorem.

When A is a Dedekind domain, not necessarily principal, and in case $P_\mu^\infty = \varnothing$, for all $\mu \in \mathbb{S}^*$, it will be possible, if necessary, to replace $\mathbb{S}$ by a suitable relatively disjoint set of superdivisors representing T. Then, some minor modifications to the proof of Theorem 1 yield

THEOREM 3. *Let A be a Dedekind domain and let P be the set of all nonzero prime ideals of A. Let $\mathbb{S}$ be a relatively disjoint set of superdivisors and let μ_0 be the GCD of any two distinct elements of $\mathbb{S}$. Suppose $\mathbb{S}$ contains at most two superdivisors μ such that $\mu_0(\mathfrak{p}) < \mu(\mathfrak{p}) = \infty$ for some $\mathfrak{p} \in P$. Then, there exists a rank two torsion-free A-module M such that $T(M) = \{\tau(\mu); \mu \in \mathbb{S}\}$.*

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