CHARACTER TABLES DETERMINE
ABELIAN SYLOW 2-SUBGROUPS

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Abstract. A finite group has an abelian \( S_2 \) if and only if every 2-element is 2-central.

In his survey talk at the AMS Summer Institute on Finite Group Theory, Santa Cruz, California, 1979, Walter Feit mentioned the following problem:

Can one read from the character table of a finite group if its Sylow \( p \)-subgroups are abelian?

The aim of this note is to show that a Sylow \( p \)-subgroup of \( G \) is abelian iff each \( p \)-element of \( G \) is \( p \)-central, provided that either \( p = 2 \) (Theorem 6) or \( G \) is \( p \)-solvable (Proposition 1). Thus in these two cases the answer to Feit's question is in the affirmative. The authors are not aware of finite groups satisfying one of the above-mentioned properties, but not the other. In this note we also show that the property: "a Sylow 2-subgroup of \( G \) is elementary abelian" can be read from the character table of \( G \) (Corollary 5).

In this note \( G \) denotes a finite group. Let \( p \) be a prime. The symbol \( S_p \) stands for "Sylow \( p \)-subgroup". An arbitrary Sylow \( p \)-subgroup of \( G \) will also be denoted by \( S_p \). An element \( x \) of \( G \) is called \( p \)-central if its centralizer contains an \( S_p \) of \( G \), and it is called real if its column in the character table of \( G \) is real. It is well known that \( x \) is real iff \( x \) is conjugate to \( x^{-1} \) in \( G \). Finally, let \( C_G^x(x) = \{ g \in G | x^g = x \text{ or } x^{-1} \} \). This is a subgroup of \( G \) and \( C_G^x(x) = C_G(x) \) unless \( x \) is a real element satisfying \( x^2 \neq 1 \), in which case \( |C_G^x(x): C_G(x)| = 2 \).

If \( G \) is \( p \)-solvable, we can easily prove the following proposition.

**Proposition 1.** Let \( G \) be a \( p \)-solvable finite group. Then \( S_p \) of \( G \) is abelian iff each \( p \)-element of \( G \) is \( p \)-central.

**Proof.** The "only if" part is trivial. So suppose that each \( p \)-element of \( G \) is \( p \)-central. By Theorem 3.3 in [3], \( G = O_{p^{mp}}^+(G) \). Now \( \overline{G} = G/O_p(G) \) is \( p \)-closed and still satisfies our assumption. Thus \( S_p \) of \( \overline{G} \) is abelian and hence \( S_p \) of \( G \) is abelian.

From now on, we shall deal with the prime \( p = 2 \) and \( G \) will denote a group of even order. The following remark is trivial, but basic.

**Proposition 2.** A nontrivial 2-central element of \( G \) is real iff it is an involution.
Proof. Every involution is real. Conversely, let \( u \neq 1 \) be a 2-central element of \( G \). Then \( C_G(u) = C_G(\bar{u}) \). Thus, if \( u \) is real, it follows that \( u \) is an involution.

Corollary 3. The 2-central involutions of \( G \) are determined by the character table of \( G \).

Corollary 4. An \( S_2 \) of \( G \) is elementary abelian iff every 2-element of \( G \) is 2-central and real.

Corollary 5. The property "an \( S_2 \) of \( G \) is elementary abelian" is determined by the character table of \( G \).

Finally, we prove

Theorem 6. Let \( G \) be a finite group. Then \( S_2 \) of \( G \) is abelian if and only if each 2-element of \( G \) is 2-central.

Proof. The "only if" part is trivial. By Proposition 1 we may assume that \( G \) is a nonsolvable group of minimal order such that each 2-element of \( G \) is 2-central, but \( S_2 \) of \( G \) is nonabelian. Clearly \( O(G) = 1 \) and \( O^2(G) = G \). If \( u \) and \( v \) are distinct involutions in \( S_2 \), then \( uv \) is a real 2-element, hence an involution by Proposition 2. Thus \( \Omega_1(S_2) \) is abelian. Moreover, it follows from our assumptions that \( F(G) = O_2(G) \) is centralized by each 2-element of \( G \), hence \( O_2(G) < Z(G) \).

Let \( E \) be the central product of all subnormal quasisimple subgroups of \( G \) and let \( F^* = O_2(G)E \). It is well known that \( C_G(F^*) = Z(F^*) \) (see [1, §10]), hence \( E \neq 1 \). Suppose that \( G \) is quasisimple. As \( \Omega_1(S_2) \) is elementary abelian every 2-element of \( G \) is 2-central, it follows by [2] that \( G \) is simple with an abelian \( S_2 \), a contradiction. So each quasisimple subnormal subgroup of \( G \) is a proper subgroup of \( G \), hence has an abelian \( S_2 \). Consequently, \( F^* = O_2(G)^*M_1 \cdot \cdots \cdot M_r \), a central product, with \( M_i \) quasisimple having abelian \( S_2 \). Thus \( F^* \) has an abelian \( S_2 \) and it suffices to show that if \( x \) is a 2-element of \( G \), then \( x \in F^* \). Suppose that \( x \not\in F^* \). Then \( x \) acts as an automorphism on \( F^* \), centralizing \( O_2(G) \) and permuting the \( M_i \). However, \( x \) centralizes an \( S_2 \) of \( F^* \), hence \( x \) normalizes each \( M_i \). Thus we may assume that \( x \) acts as an outer automorphism of even order on some \( M_i \) and hence also on the simple group \( M = M_i/Z(M_i) \) [1, Lemma 10.3]. Since \( M \) is a simple group with an abelian \( S_2 \) and with an outer automorphism of even order, it follows by [4] that \( M = PSL(2, q) \), with \( q = 2^n > 2 \) or \( q \equiv 3 \) or \( 5 \) (mod 8), \( q > 5 \). If \( q = 2^n \) and the order of \( x \) as an outer automorphism is \( 2^k = r \), then \( C_M(x) = PSL(2, 2^{n/r}) \) and \( x \) does not centralize an \( S_2 \) of \( M \), a contradiction. If \( q \equiv 3 \) or \( 5 \) (mod 8) and \( q > 5 \), then \( Z(M_i) = 1 \), and \( x \) acts on \( M_i \) as an element of \( PGL(2, q) \), hence it does not centralize an \( S_2 \) of \( M_i \), a contradiction. The proof of Theorem 6 is complete.

Corollary 7. The property "an \( S_2 \) of \( G \) is abelian" is determined by the character table of \( G \).

Added in Proof. The authors have been informed that \( ^2F_4(2) \) has just one class of 3-elements and its \( S_3 \) is nonabelian of order 27 and exponent 3.
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