SEMILOCAL SKEW GROUP RINGS

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Abstract. Semilocal skew group rings \( R \star_{\theta} G \) are investigated. The full characterization is given in the case of algebras over a field of characteristic zero. The relationship between semilocal skew group rings and semilocal ordinary group rings \( R[G] \) is considered.

In the present paper all rings are assumed to have a unity element. Let \( R \) be a ring, \( G \) be a group and \( \theta: G \to \text{Aut}(R) \) a group homomorphism. By the skew group ring \( R \star_{\theta} G \) we shall mean \( \bigoplus_{g \in G} Rg \) with addition given componentwise and multiplication given by formula \( (rg)(sh) = rs^{\theta(g)h}g \) for \( r, s \in R, g, h \in G \). If \( \theta \) is trivial then we get ordinary group ring \( R[G] \). The full characterization of semilocal group rings in the case of algebras over a field of characteristic zero is given in [2]. On the other hand, semilocal skew group rings in the case when \( \theta \) is injective and \( R \) is a field were characterized in [5].

First, we shall prove a useful characterization of semilocal rings. Let \( A \) be a ring, \( x \in A \). The following sequence of elements was used in [7]: \( f_1(x) = x, f_i(x) = f_{i-1}(x)(1 - f_{i-1}(x)) \) for \( i > 1 \). We shall say that \( A \) is \( W_n \)-ring if for any \( x \in A \) there exists \( i, 1 < i < n \), such that \( 1 - f_i(x) \) is invertible in \( A \).

Lemma 1. Let \( A \) be a primitive ring. If \( A \) satisfies \( W_n \) for some \( n \), then \( A \simeq M_t(D) \) with \( t < 2^n - 2 \).

Proof. Let \( f(x) = (f_1(x) - 1)(f_2(x) - 1) \cdots (f_n(x) - 1) \). Since \( \deg f_i = 2^{i-1} \) we see that \( \deg f = 1 + 2 + \cdots + 2^{n-1} = 2^n - 1 = m \).

Let \( V \) be the faithful irreducible \( A \)-module with commuting ring \( D \). We claim \( \dim_D V < m \). If not, we may choose \( v_0, v_1, \ldots, v_{m-1} \in V \) which are \( D \)-linearly independent and put \( f(x) = x^m - \sum_{i=0}^{m-1} a_i x^i \). By the Jacobson density theorem [1], there exists \( r \in A \) with \( v_i r = v_{i+1} \) for \( i = 0, 1, \ldots, m - 2 \) and \( v_{m-1} r = \sum_{i=0}^{m-1} a_i v_i \). Thus \( v_0 r^i = v_i \) for \( i < m - 1 \) and \( v_0 r^m = v_{m-1} r = \sum_{i=0}^{m-1} a_i v_i = v_0 \sum_{i=0}^{m-1} a_i r^i \). In other words, \( v_0 f(r) = 0 \) but certainly no monic polynomial in \( r \) of degree \( < m \) can annihilate \( v_0 \). Now \( f(r) = \Pi(f_j(r) - 1) \) so if some \( f_j(r) - 1 \) is invertible, then \( v_0 \Pi_{j \neq j}(f_j(r) - 1) = 0 \), a contradiction. Thus \( f_j(r) - 1, j = 1, 2, \ldots, n \), is invertible and \( A \) does not satisfy \( W_n \), a contradiction. Hence \( \dim_D V < m \) and \( A \simeq M_t(D) \) with \( t < m - 1 = 2^n - 2 \) [1].

Lemma 2. Let \( A \) be a ring. Then \( A \) is semilocal if and only if \( A \) satisfies \( W_n \) for some \( n \) and has only finitely many maximal ideals (2-sided).

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PROOF. Necessity follows from the definition of semilocal rings and from [7].
Let $A$ be a $W_n$-ring and $P$ a primitive ideal in $A$. Then $A/P$ satisfies $W_n$ and it follows from Lemma 1 that it is simple and artinian. Thus $P$ is maximal. If $A$ has only finitely many maximal ideals then $A$ is semilocal.

Now, if $B \subseteq A$ is a subring with the same unity and $B$ is a direct summand of the left $B$-module $A$ then we shall write $B|A_B$ (cf. [6, Chapter 7]).

It is well known that if $B|A_B$ then
(i) any element of $B$ invertible in $A$ is invertible in $B$,
(ii) $J(A) \cap B \subset J(B)$,
(iii) if $I$ is any right ideal of $B$ then $IA \cap B = I$.

**Lemma 3.** Let $A$ be a ring and $B \subseteq A$ be a subring such that $B|A_B$. If $A$ is semilocal then so is $B$.

**Proof.** We know $A$ satisfies $W_n$ for some $n$. Since any element of $B$ invertible in $A$ is invertible in $B$, it follows that $B$ satisfies $W_n$. Let us suppose $M_1, M_2, \ldots$ are infinitely many maximal ideals of $B$ and set $I_i = M_1 \cap M_2 \cap \cdots \cap M_i$. Then we know for some $j$ that $I_j \subseteq I_jA \subseteq J(A) + I_{j+1}A$. Further since $I_j/I_{j+1} \cong (I_j + M_{j+1})/I_{j+1} = B/I_{j+1}$ we see that there exists $x \in I_j$ with $x \equiv 1$ mod $I_{j+1}$. We can write $x = k + b$ with $k \in J(A), b \in I_{j+1}A$. Then $k - 1 = (x - 1) - b \in I_{j+1}A$. But $k - 1$ is invertible and so $I_{j+1}A = A$. Since $I_{j+1}A \cap B = I_{j+1}$, this is a contradiction. Thus $B$ has only finitely maximal ideals and it is semilocal by Lemma 2.

**Theorem 1.** Let $R * \theta G$ be semilocal. Then
(1) $R * H$ is semilocal for every subgroup $H$ in $G$,
(2) $B * \theta G$ is semilocal for every subring $B$ in $R$ with $B|A_B, B^{\theta(G)} = B$ where $\sigma: G \rightarrow \text{Aut}(B)$ is induced by $\theta$.

**Proof.** It follows from [5] that in both cases the considered subring of $R * \theta G$ satisfies the assumptions of Lemma 3 and hence it is semilocal.

Since Wood's proof that $G$ is torsion holds for skew group rings we then obtain

**Corollary 1.** If $R * \theta G$ is semilocal then $R$ is semilocal and $G$ is torsion.

Now, the following result follows from [5].

**Corollary 2.** Let $G$ be finite. Then $R * \theta G$ is semilocal if and only if $R$ is semilocal.

It is easy to check that the above corollaries at least hold for crossed products.

**Theorem 2.** Let $K$ be a field of characteristic zero and $R$ be a $K$-algebra. Then the following conditions are equivalent:
(1) $R * \theta G$ is semilocal,
(2) $R$ is semilocal and $G$ is finite.

**Proof.** (1) $\Rightarrow$ (2). It follows from Theorem 1 that $Q[G]$ is semilocal where $Q$ is the field of rationals. Hence $G$ is finite by [2].

(2) $\Rightarrow$ (1) follows from Corollary 2.
It is easy to check that in the above theorem it is enough to assume that the additive group of the ring \( R/J(R) \) is not torsion. Moreover, implication (1) \( \Rightarrow \) (2) is a direct consequence of a spectral characterization of semilocal algebras over infinite fields [3].

As we have seen, in some cases, the ring \( R \ast \theta G \) is semilocal if and only if the ordinary group ring \( R[G] \) is semilocal.

**Proposition.** Let \( K \) be a field. If \( K \ast \theta G \) is semilocal then so is \( K[G] \).

**Proof.** Let \( H = \ker \theta \). Then \( K[H] \) is semilocal by Theorem 1 and \( K \ast \theta G/H \) is semilocal since it is a homomorphic image of \( K \ast \theta G \). Here \( \theta : G/H \rightarrow \text{Aut}(K) \) is the injective homomorphism induced by \( \theta \). Thus, it follows from [5] that \( G/H \) is finite. Now, the result follows easily from the fact that \( K[G] \) has a normalizing basis over \( K[H] \) [6].

We propose the following

**Conjecture.** If \( R \ast \theta G \) is semilocal then so is \( R[G] \), at least in the case when \( J(R) = 0 \).

It is conjectured that \( K[G] \) is semilocal if and only if \( G \) contains a \( p \)-subgroup \( H \) of finite index with \( J(K[H]) = \omega(K[H]) \) where \( \text{char } K = p > 0 \) [6]. It was proved in some cases in [4]. However, the converse of the proposition does not hold even if \( G \) satisfies much stronger conditions.

**Example.** Let \( G \) be an infinite elementary abelian \( p \)-group and let \( K \) be a field of characteristic \( p \) acted upon faithfully by \( G \). Then \( K \ast G \) is not semilocal [5], but \( K[G] \) is.

**References**


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