PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 4, December 1980

SEMILOCAL SKEW GROUP RINGS

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ABSTRACT. Semilocal skew group rings $R *_\theta G$ are investigated. The full characterization is given in the case of algebras over a field of characteristic zero. The relationship between semilocal skew group rings and semilocal ordinary group rings $R[G]$ is considered.

In the present paper all rings are assumed to have a unity element. Let R be a ring, G be a group and θ: $G \to \mathrm{Aut}(R)$–a group homomorphism. By the skew group ring $R *_\theta G$ we shall mean $\oplus\sum_{g\in G} Rg$ with addition given componentwise and multiplication given by formula $(rg)(sh) = rs^{\theta(g)}gh$ for $r, s \in R$, g, $h \in G$. If θ is trivial then we get ordinary group ring $R[G]$. The full characterization of semilocal group rings in the case of algebras over a field of characteristic zero is given in [2]. On the other hand, semilocal skew group rings in the case when θ is injective and R is a field were characterized in [5].

First, we shall prove a useful characterization of semilocal rings. Let A be a ring, $x \in A$. The following sequence of elements was used in [7]: $f_1(x) = x$, $f_i(x) = f_{i-1}(x)(1 - f_{i-1}(x))$ for $i > 1$. We shall say that A is W_n-ring if for any $x \in A$ there exists i, $1 \leqslant i \leqslant n$, such that $1 - f_i(x)$ is invertible in A.

LEMMA 1. *Let A be a primitive ring. If A satisfies W_n for some n, then $A \simeq M_t(D)$ with $t \leqslant 2^n - 2$.*

PROOF. Let $f(x) = (f_1(x) - 1)(f_2(x) - 1) \cdots (f_n(x) - 1)$. Since $\deg f_i = 2^{i-1}$ we see that $\deg f = 1 + 2 + \cdots + 2^{n-1} = 2^n - 1 = m$.

Let V be the faithful irreducible A-module with commuting ring D. We claim $\dim_D V < m$. If not, we may choose $v_0, v_1, \ldots, v_{m-1} \in V$ which are D-linearly independent and put $f(x) = x^m - \sum_0^{m-1} a_i x^i$. By the Jacobson density theorem [1], there exists $r \in A$ with $v_i r = v_{i+1}$ for $i = 0, 1, \ldots, m-2$ and $v_{m-1} r = \sum_0^{m-1} a_i v_i$. Thus $v_0 r^i = v_i$ for $i \leqslant m-1$ and $v_0 r^m = v_{m-1} r = \sum_0^{m-1} a_i v_i = v_0 \sum_0^{m-1} a_i r^i$. In other words, $v_0 f(r) = 0$ but certainly no monic polynomial in r of degree $< m$ can annihilate v_0. Now $f(r) = \prod(f_i(r) - 1)$ so if some $f_j(r) - 1$ is invertible, then $v_0 \prod_{i \neq j}(f_i(r) - 1) = 0$, a contradiction. Thus no $f_j(r) - 1$, $j = 1, 2, \ldots, n$, is invertible and A does not satisfy W_n, a contradiction. Hence $\dim_D V < m$ and $A \simeq M_t(D)$ with $t \leqslant m - 1 = 2^n - 2$ [1].

LEMMA 2. *Let A be a ring. Then A is semilocal if and only if A satisfies W_n for some n and has only finitely many maximal ideals* (*2-sided*).

Received by the editors November 20, 1979 and, in revised form, February 1, 1980.
AMS (*MOS*) *subject classifications* (1970). Primary 16A24, 16A46.

PROOF. Necessity follows from the definition of semilocal rings and from [7].

Let A be a W_n-ring and P a primitive ideal in A. Then A/P satisfies W_n and it follows from Lemma 1 that it is simple and artinian. Thus P is maximal. If A has only finitely many maximal ideals then A is semilocal.

Now, if $B \subset A$ is a subring with the same unity and B is a direct summand of the left B-module A then we shall write ${}_B B|_B A$ (cf. [6, Chapter 7]).

It is well known that if ${}_B B|_B A$ then

(i) any element of B invertible in A is invertible in B,

(ii) $J(A) \cap B \subset J(B)$,

(iii) if I is any right ideal of B then $IA \cap B = I$.

LEMMA 3. *Let A be a ring and $B \subset A$ be a subring such that ${}_B B|_B A$. If A is semilocal then so is B.*

PROOF. We know A satisfies W_n for some n. Since any element of B invertible in A is invertible in B, it follows that B satisfies W_n. Let us suppose $M_1, M_2, \ldots$ are infinitely many maximal ideals of B and set $I_i = M_1 \cap M_2 \cap \cdots \cap M_i$. Then we know for some j that $I_j \subset I_j A \subset J(A) + I_{j+1}A$. Further since $I_j/I_{j+1} \simeq (I_j + M_{j+1})/I_{j+1} = B/I_{j+1}$ we see that there exists $x \in I_j$ with $x \equiv 1 \bmod I_{j+1}$. We can write $x = k + b$ with $k \in J(A)$, $b \in I_{j+1}A$. Then $k - 1 = (x - 1) - b \in I_{j+1}A$. But $k - 1$ is invertible and so $I_{j+1}A = A$. Since $I_{j+1}A \cap B = I_{j+1}$, this is a contradiction. Thus B has only finitely maximal ideals and it is semilocal by Lemma 2.

THEOREM 1. *Let $R *_\theta G$ be semilocal. Then*

(1) $R *_\theta H$ is semilocal for every subgroup H in G,

(2) $B *_\sigma G$ is semilocal for every subring B in R with ${}_B B|_B A$, $B^{\theta(G)} = B$ where σ: $G \to \mathrm{Aut}(B)$ is induced by θ.

PROOF. It follows from [5] that in both cases the considered subring of $R *_\theta G$ satisfies the assumptions of Lemma 3 and hence it is semilocal.

Since Wood's proof that G is torsion holds for skew group rings we then obtain

COROLLARY 1. *If $R *_\theta G$ is semilocal then R is semilocal and G is torsion.*

Now, the following result follows from [5].

COROLLARY 2. *Let G be finite. Then $R *_\theta G$ is semilocal if and only if R is semilocal.*

It is easy to check that the above corollaries at least hold for crossed products.

THEOREM 2. *Let K be a field of characteristic zero and R be a K-algebra. Then the following conditions are equivalent:*

(1) $R *_\theta G$ is semilocal,

(2) R is semilocal and G is finite.

PROOF. (1) $\Rightarrow$ (2). It follows from Theorem 1 that $Q[G]$ is semilocal where Q is the field of rationals. Hence G is finite by [2].

(2) $\Rightarrow$ (1) follows from Corollary 2.

It is easy to check that in the above theorem it is enough to assume that the additive group of the ring $R/J(R)$ is not torsion. Moreover, implication (1) $\Rightarrow$ (2) is a direct consequence of a spectral characterization of semilocal algebras over infinite fields [3].

As we have seen, in some cases, the ring $R *_\theta G$ is semilocal if and only if the ordinary group ring $R[G]$ is semilocal.

PROPOSITION. *Let K be a field. If $K *_\theta G$ is semilocal then so is $K[G]$.*

PROOF. Let $H = \ker \theta$. Then $K[H]$ is semilocal by Theorem 1 and $K *_{\bar\theta} G/H$ is semilocal since it is a homomorphic image of $K *_\theta G$. Here $\bar\theta$: $G/H \to \mathrm{Aut}(K)$ is the injective homomorphism induced by θ. Thus, it follows from [5] that G/H is finite. Now, the result follows easily from the fact that $K[G]$ has a normalizing basis over $K[H]$ [6].

We propose the following

CONJECTURE. *If $R *_\theta G$ is semilocal then so is $R[G]$, at least in the case when* $J(R) = 0$.

It is conjectured that $K[G]$ is semilocal if and only if G contains a p-subgroup H of finite index with $J(K[H]) = \omega(K[H])$ where char $K = p > 0$ [6]. It was proved in some cases in [4]. However, the converse of the proposition does not hold even if G satisfies much stronger conditions.

EXAMPLE. Let G be an infinite elementary abelian p-group and let K be a field of characteristic p acted upon faithfully by G. Then $K * G$ is not semilocal [5], but $K[G]$ is.

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