PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 4, December 1980

ASYMPTOTIC PRIME DIVISORS AND ANALYTIC SPREADS

STEPHEN McADAM[1]

ABSTRACT. Let I be an ideal in a Noetherian domain R, and let $\hat{I}$ be the integral closure of I. Let $\hat{A}^*(I) = \operatorname{Ass}(R/\hat{I}^n)$ for n large (it being known that for large n this set does not vary with n). Suppose that R satisfies the altitude formula. Then it is shown that $P \in \hat{A}^*(I)$ if and only if height $P = l(I_P)$, the analytic spread of I_P.

Introduction. Let I be an ideal in a Noetherian ring. For $n \geq 1$, let $A(n)$ be the set of prime divisors of I^n, $A(n) = \operatorname{Ass}(R/I^n)$. A recent paper of Brodmann [1] shows that $A(n)$ is constant for n large. In [5] that constant is denoted $A^* = A^*(I)$. In general it is difficult to explicitly determine A^* for a given ideal I, although in [5, Corollary 22] this is done for R a 2-dimensional normal domain. This paper will discuss a concept related to A^*, namely $\hat{A}^*$. Let $\hat{I}$ denote the integral closure of the ideal I, and let $\hat{A}(n) = \operatorname{Ass}(R/\hat{I}^n)$, the prime divisors of $\hat{I}^n$. If height $I \geq 1$, [5, Proposition 7] shows that $\hat{A}(n)$ is constant for large n. That constant is denoted by $\hat{A}^* = \hat{A}^*(I)$. The purpose of this paper is to characterize $\hat{A}^*$ for any ideal I in a Noetherian domain satisfying the altitude formula. The characterization is $P \in \hat{A}^*$ if and only if height $P = l(IR_P)$, the analytic spread of IR_P.

Preliminaries. Throughout this paper, R will denote a Noetherian domain, I an ideal of R, and P a prime ideal of R containing I. The domain T will always be $T = R[Ix] = R + Ix + I^2x^2 + \dots$, x an indeterminate. Since $T \subset R[x]$, obviously the transcendence degree of T over R is 1. We will occasionally mention the form ring of I, $R/I + I/I^2 + \dots$. Note that this is isomorphic to T/IT. If (R, P) is local, we will also use the ring $R/P + I/PI + I^2/PI^2 + \dots$, which is isomorphic to T/PT. Finally, P'' will be $P + Ix + I^2x^2 + \dots$ in T.

If (R, P) is a local domain and I is an ideal of R, then $l(I)$ denotes the analytic spread of I. Recall that there are various characterizations of $l(I)$. (i) If R/P is infinite and if J is a minimal reduction of I then $l(I) = v(J)$, the minimal number of generators of J. (ii) $l(I) = \operatorname{height}(P''/PT)$. (See [7] and [8] for basics on reductions and $l(I)$.) Also by the altitude inequality (stated below) height $P + \operatorname{TRD}(T/R) \geq$ height $P'' + \operatorname{TRD}(P''/P)$ giving height $P + 1 \geq$ height $P'' \geq \operatorname{height}(P''/PT) = l(I)$. Thus height $P \geq l(I)$. (See [2] for more.)

Received by the editors July 10, 1979.

AMS (MOS) subject classifications (1970). Primary 13E05; Secondary 13A15.

Key words and phrases. Noetherian domain, prime divisors, analytic spread, integral closure of an ideal.

[1]Partially supported by the National Science Foundation.

Let the domain S be a finitely generated ring extension of R. Let Q be prime in S with $Q \cap R = P$. It is well known that height P + TRD$(S/R) \geq$ height Q + TRD(Q/P). Here "TRD" denotes transcendence degree and TRD(Q/P) refers to the transcendence degree of S/Q over R/P. If in fact height P + TRD(S/R) = height Q + TRD(Q/P) for all such S and Q, then R is said to *satisfy the altitude formula*. Almost all known Noetherian domains do satisfy the altitude formula. The only known counterexamples are variations on [6, Example 2, pp. 202–205]. Thus assuming that the altitude formula holds is a minor restriction.

$\hat{A}^*$ and $l(I)$. Our first lemma is essentially a restatement of [5, Proposition 18] in a more efficient manner.

LEMMA 1. *Let R be a Noetherian domain which satisfies the altitude formula. Let $0 \neq I \subseteq P$ be ideals of R, with P prime. Then $P \in \hat{A}^*$ if and only if there is a height one prime P' of $T = R + Ix + I^2x^2 + \dots$, with $P' \cap R = P$. If this is the case, then P' is homogeneous.*

PROOF. Suppose first that such a P' exists. As $I \subseteq P \subset P'$, $IT \subseteq P'$ and so in the form ring of I, T/IT, P'/IT is a minimal prime. According to [5, Proposition 18], in order to show that $P \in \hat{A}^*$ we need only show that P'/IT is a relevant prime. Being a minimal prime, P'/IT is homogeneous (thus P' is homogeneous as stated), and so if it is not relevant then clearly $P' = P + Ix + I^2x^2 + \dots$, so that $T/P' = R/P$. Applying the altitude formula to $R \subset T$ and the primes P and P' gives height P + TRD(T/R) = height P' + TRD(P'/P), that is, height $P + 1 = 1 + 0$. Thus height $P = 0$ contradicting that $0 \neq I \subseteq P$. Therefore P'/IT is relevant, as required.

Conversely, suppose that $P \in \hat{A}^*$. By [5, Proposition 18], in the form ring T/IT there is a minimal prime, call it P'/IT, with $(P'/IT) \cap (R/I) = P/I$. To prove the lemma, we must only show that in T, height $P' = 1$. We go to the Rees ring $T + x^{-1}R[x^{-1}]$, and consider the prime $P' + x^{-1}R[x^{-1}]$. Since T, being a finitely generated extension of R, satisfies the altitude formula, and since $T/P' = (T + x^{-1}R[x^{-1}])/(P' + x^{-1}R[x^{-1}])$ we have height P' = height $P' + x^{-1}R[x^{-1}]$. As P' is minimal over IT, $P' + x^{-1}R[x^{-1}]$ is minimal over $IT + x^{-1}R[x^{-1}] = x^{-1}(T + x^{-1}R[x^{-1}])$, which is a principal ideal of the Rees ring. Accordingly, height $P' = 1$.

COROLLARY 2. *Let (R, P) be a local domain satisfying the altitude formula. Let I be an ideal of R. Then $P \in \hat{A}^*$ if and only if PT is a height one ideal of T.*

PROOF. If $P \in \hat{A}^*$, pick P' as in the lemma. Obviously $PT \subseteq P'$ and so height $PT = 1$. Conversely if height $PT = 1$, let P' be a height one prime of T containing PT. Thus $P \subset PT \subseteq P'$ and so $P' \cap R = P$. By the lemma, $P \in \hat{A}^*$.

THEOREM 3. *Let R be a Noetherian domain satisfying the altitude formula. Let $I \neq 0$ be an ideal of R and let P be a prime containing I. Then $P \in \hat{A}^*$ if and only if $l(IR_P)$ = height P.*

Proof. We may assume that R is local at P, and write $l(I)$ for $l(IR_P)$. Suppose first that height $P = l(I)$, call this n. Let P'' be the prime $P + Ix + I^2x^2 + \ldots$ of $T = R + Ix + I^2x^2 + \ldots$. Since $n = l(I) = \text{height}(P''/PT)$ in the ring T/PT, we have a chain of primes $P_0' \subset P_1' \subset \cdots \subset P_n' = P''$ in T with $PT \subseteq P_0'$. Obviously $P_0' \cap R = P$. In order to show that $P \in \hat{A}^*$, in view of the lemma we must only show that height $P_0' = 1$. We apply the altitude formula to $R \subset T$ and the primes P and P''. Since height $P = n$, $\text{TRD}(T/R) = 1$, and $T/P'' = R/P$, the altitude formula yields $n + 1 =$ height P''. Now the chain $P_0' \subset P_1' \subset \cdots \subset P_n' = P''$ shows that height $P_0' = 1$ as required.

Conversely, suppose that $P \in \hat{A}^*$, and now let $n = l(I)$. As above we have $P_0' \subset P_1' \subset \cdots \subset P_n' = P''$ with $PT \subseteq P_0'$. Since $P \in \hat{A}^*$, by Lemma 1 there is a height 1 homogeneous prime P' of T with $P' \cap R = P$. As (R, P) is local, and P' is homogeneous, $P' \subseteq P + Ix + I^2x^2 + \cdots = P''$. Let $k = \text{height}(P''/P')$. As $P \subset P'$, $PT \subseteq P'$. Thus $k = \text{height}(P''/P') \leq \text{height}(P''/PT) = l(I) = n$. That is $k \leq n$. As R satisfies the altitude formula, T is catenary [4, Corollary 2.5] and so height $P'' = \text{height}(P''/P') +$ height $P' = k + 1$. Also height $P'' = \text{height}(P''/P_0') +$ height $P_0' \geq n + 1$ (by the existence of the chain $P_0' \subset \cdots \subset P_n' = P''$ and the fact that $0 \neq PT \subset P_0'$). Thus we have $k + 1 =$ height $P'' \geq n + 1$. As we previously saw $k \leq n$ we get $n = k$ and height $P'' = n + 1$. Finally the altitude formula applied to $R \subset T$ and P and P'' gives height $P + 1 =$ height $P'' + 0$ so that height $P = n = l(I)$.

Corollary 4. *Let R be a Noetherian domain satisfying the altitude formula. Let $0 \neq I \subseteq P$ be ideals of R with P prime. If $P \in \hat{A}(n)$ for any $n \geq 1$, then* height $P = l(IR_P)$.

Proof. By [9, Theorem 2.5], $\hat{A}(n) \subseteq \hat{A}^*$, and so the corollary is immediate from Theorem 3.

We can strengthen Corollary 2 in the case that I is basic. Recall that an ideal in a local domain is *basic* if $v(I) = l(I)$, or equivalently if I is a minimal reduction of itself. (Notice that in discussing $\hat{A}^*$, one may always assume that I is basic, since if J is a minimal reduction of I, making J basic, then for all $n \geq 1$, J^n reduces I^n so that $\hat{J^n} = \hat{I^n}$.)

Corollary 5. *Let I be a basic ideal in a local domain (R, P) which satisfies the altitude formula. Then $P \in \hat{A}^*$ if and only if PT is a height* 1 *prime of T.*

Proof. Assume that $P \in \hat{A}^*$. We refer to the second half of the proof of Theorem 3. We have $\text{height}(P''/P') = k = n = l(I) = v(I)$. If $I = (a_1, \ldots, a_n)$ then $T = R[a_1x, \ldots, a_nx]$ and we have an obvious homomorphism from $R[x_1, \ldots, x_n]$ onto T. Let Q'' and Q' be the inverse images of P'' and P' respectively. Since $P'' \cap R = P = P' \cap R$, we have $Q'' \cap R = P = Q' \cap R$. Thus Q'' and Q' are two primes in $R[x_1, \ldots, x_n]$ both lying over P. However $\text{height}(Q''/Q') = \text{height}(P''/P') = n$. This forces Q' to be $PR[x_1, \ldots, x_n]$ and so its image P' is PT. Thus PT is a height 1 prime of T. The converse follows from Corollary 2.

THEOREM 6. *Let R be a 2-dimensional normal Noetherian domain. Then for any ideal I of R, $A^* = \hat{A}^*$.*

PROOF. By [9, Corollary 2.6.1], $\hat{A}^* \subseteq A^*$. Conversely, suppose that $P \in A^*$. If P is minimal over I then obviously $P \in \hat{A}^*$. If P is not minimal over I then height $P = 2$. By [5, Proposition 21], IR_P is not principal. I claim that $l(IR_P) > 1$. If $l(IR_P) = 1$ then by the usual method, we may assume R_P/P_P is infinite, so for some $a \in R_P$, $\widehat{aR_P}$ reduces IR_P. Thus $aR_P \subseteq IR_P \subseteq a\hat{R}_P$. However, since R_P is normal, $aR_P = aR_P$ showing that IR_P is principal. This contradiction shows that $l(IR_P) > 1$. We know $l(IR_P) \leqslant$ height $P = 2$. Thus $l(IR_P) = 2 =$ height P. By Theorem 3, $P \in \hat{A}^*$, since 2-dimensional normal Noetherian domains are Cohen-Macaulay and hence satisfy the altitude formula [6, 35.5].

COROLLARY 7. *Let I be an ideal in a 2-dimensional normal Noetherian domain R. Let P be prime in R containing I. Then $P \in A^*$ if and only if* height $P = l(IR_P)$.

PROOF. Immediate from Theorems 3 and 6.

Corollary 7 fails without normality. If R is not normal then there will always be elements $a \in R$ for which $(a) \subsetneqq (\hat{a})$. In our next proposition we use this to find an I for which $A^* \neq \hat{A}^*$.

PROPOSITION 8. *Let (R, P) be a local domain with* $\dim R > 1$. *Let $0 \neq a \in R$ and suppose that $y \in (\hat{a}) - (a)$. Let $I = (Py, a)$. Then for all $n \geqslant 1$, $P \in A(n)$. If (R, P) satisfies the altitude formula, then for all $n \geqslant 1$, $P \in A(n) - \hat{A}(n)$.*

PROOF. As $y \in (\hat{a})$, y satisfies an equation $y^m + r_1 a y^{m-1} + \cdots + r_m a^m = 0$. Note $m > 1$ since $y \notin (a)$. Suppose that here m is the least possible. If $1 \leqslant n < m$, I claim that $y^n \notin I^n$ for if $y^n \in I^n = (Py, a)^n$, write $y^n = r_0 y^n p_n + r_1 y^{n-1} p_{n-1} a + \cdots + r_n p_0 a^n$ with $p_i \in P^i$. Thus $y^n(1 - r_0 p_n) - r_1 a p_{n-1} y^{n-1} - \cdots - r_n p_0 a^n = 0$. This is impossible since $1 - r_0 p_n$ is a unit and $n < m$. Thus $y^n \notin I^n$ for $1 \leqslant n < m$. Now $P^n y^n \subseteq (Py, a)^n = I^n$, and so P^n consists of zero divisors modulo I^n. Thus $P \in A(n)$ for $1 \leqslant n < m$.

I now claim that $aI^{m-1} = I^m$. Obviously $I^m = (Py)^m + a(Py)^{m-1} + \cdots + a^{m-1}Py + (a)^m$, and each term of this sum is contained in aI^{m-1} except the term $(Py)^m$. However we have $y^m + r_1 a y^{m-1} + \cdots + r_m a^m = 0$ from which we see that $(Py)^m \subseteq aI^{m-1}$. Thus $I^m \subseteq aI^{m-1}$. The other inclusion holds since $a \in I$.

Now consider $n \geqslant m$. By the first paragraph of this proof, we already have $P = (I^{m-1}: c)$ for some $c \in R$. Obviously $P = (I^{m-1}a^{n-m+1}: ca^{n-m+1})$. However since $aI^{m-1} = I^m$, $a^{n-m+1}I^{m-1} = I^n$, and so $P \in A(n)$ for all $n \geqslant 1$.

Finally suppose that R satisfies the altitude formula. Since height $P = \dim R > 1$, in order to show that $P \notin \hat{A}(n)$ for all $n \geqslant 1$, in view of Corollary 4, it is enough to show that $l(I) = 1 <$ height P. However the second paragraph of the proof shows that (a) reduces I. Thus $l(I) = 1$ as desired.

REFERENCES

1. M. Brodmann, *Asymptotic stability of* Ass(R/I^n), Proc. Amer. Math. Soc. **74** (1979), 16–18.

2. ______, *The asymptotic nature of the analytic spread* (preprint).

3. I. Kaplansky, *Commutative rings*, rev. ed., Univ. of Chicago Press, Chicago, Ill., 1974.

4. S. McAdam and E. Davis, *Prime divisors and saturated chains*, Indiana Univ. Math. J. **26** (1977), 653–662.

5. S. McAdam and P. Eakin, *The asymptotic* Ass, J. Algebra **61** (1979), 71–81.

6. M. Nagata, *Local rings*, Interscience, New York, 1962.

7. D. G. Northcott and D. Rees, *Reductions of ideals in local rings*, Proc. Cambridge Philos. Soc. **50** (1954), 145–158.

8. ______, *A note of reductions of ideals with an application to the generalized Hilbert function*, Proc. Cambridge Philos. Soc. **50** (1954), 353–359.

9. L. J. Ratliff, Jr., *On prime divisors of* I^n*, n large*, Michigan Math. J. **23** (1976), 337–352.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS, AUSTIN, TEXAS 78712