ON BIALGEBRAS WHICH ARE SIMPLE HOPF MODULES

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Abstract. This paper gives a module characterization of commutative or cocommutative Hopf algebras over a field.

0. Introduction. Let $A$ be a bialgebra over a field $k$. Then $A$ has a natural left $A$-Hopf module structure, and if $A$ is a Hopf algebra, an easy calculation with the antipode shows that $A$ is a simple Hopf module. We show that a commutative or cocommutative bialgebra over a field $k$ which is a simple left Hopf module is a Hopf algebra. From this result we derive a module-theoretic characterization of commutative or cocommutative bialgebras over a field $k$ which are Hopf algebras; namely such a bialgebra is a Hopf algebra if and only if all left $A$-Hopf modules are free (or (0)).

Generally a commutative or cocommutative bialgebra $A$ over a field $k$ has a unique maximal subcoalgebra $A^{(1)}$ which is a Hopf algebra. In both cases $A^{(1)}$ can be described in terms of grouplike elements—the basic results of this paper are derivatives of elementary observations concerning grouplikes in certain bialgebras.

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1. Preliminaries. In this section we show that any bialgebra $A$ over a field $k$ has a unique subcoalgebra $A^{(1)}$ maximal among the subcoalgebras $D$ such that the inclusion $i_D \in \text{Hom}(D, A)$ has an inverse in the convolution algebra. We will see that $A^{(1)}$ is characterized by its simple subcoalgebras.

For a coalgebra $C$ over $k$ recall that the wedge product $U \wedge V$ of subspaces $U, V \subseteq C$ is defined by $U \wedge V = \Delta^{-1}(U \otimes C + C \otimes V)$. The wedge product of subcoalgebras is a subcoalgebra.

Lemma 1. Let $C$ be a coalgebra over a field $k$, and suppose $E, D', D'' \subseteq C$ are subcoalgebras, $E$ simple.

(a) If $E \subseteq \sum D$, where $D$ runs over a family of subcoalgebras of $C$, then $E \subseteq D$ for some $D$.

(b) If $E \subseteq D' \wedge D''$ then $E \subseteq D'$ or $E \subseteq D''$.
(c) If $A$ is a bialgebra and $E \subseteq D'D''$, then $E \subseteq E'E''$ where $E' \subseteq D'$ and $E'' \subseteq D''$ are simple subcoalgebras.

(d) If $f: C' \to C$ is a surjective coalgebra map, then $E \subseteq f(E')$ for some simple subcoalgebra $E' \subseteq C'$.

Proof. (a) is [S, Proposition 8.0.3.a]. To show (b) note that $\Delta E \subseteq D' \otimes C + C \otimes D''$ means $(E \otimes E) \cap (D' \otimes C + C \otimes D'') \neq (0)$. By (a) if $U$ is a simple subcoalgebra of this intersection, then $U \subseteq D' \otimes C$ or $U \subseteq C \otimes D''$, so $E \subseteq D'$ or $E \subseteq D''$. To show (d) observe that $C_0 \subseteq f(C_0)$ by [HR, 2.3.9]. Thus writing $C' = \Sigma E'$ as the (direct) sum of the simple subcoalgebras of $C'$, we have $E \subseteq \Sigma f(E')$, and hence (d) follows by (a). To show (c), first observe that $E \subseteq m(U)$ for some simple subcoalgebra $U \subseteq D' \otimes D''$ by (d), where $m: D' \otimes D'' \to D'D''$ is multiplication. By [HR, 2.3.13] $(D' \otimes D'')_0 \subseteq D'_0 \otimes D''_0$, so $U \subseteq E' \otimes E''$ for simple subcoalgebras $E' \subseteq D'$ and $E'' \subseteq D''$ by (a). Thus $E \subseteq E'E''$. Q.E.D.

For a coalgebra $C$ and an algebra $A$ over $k$ recall that the unity of the convolution algebra $\text{Hom}(C, A)$ is given by $\epsilon(c) = \epsilon(c)1_A$ and the product is $f * g(c) = \Sigma f(c(1))g(c(2))$ for $c \in C$ and $f, g \in \text{Hom}(C, A)$. The following is a refinement of [T, Lemma 14].

Lemma 2. Let $C$ be a coalgebra and $A$ be an algebra over a field $k$, and let $x \in \text{Hom}(C, A)$.

(a) If $x \equiv 0$ on $C_0$ then $X \mapsto x$ determines an algebra map $k[[X]] \to \text{Hom}(C, A)$. (Thus $x$ is invertible if $x \equiv e$ on $C_0$.)

(b) $x$ is left (resp. right) invertible if and only if $x|_E \in \text{Hom}(E, A)$ is left (resp. right) invertible for all simple subcoalgebras $E \subseteq C$. (Hence $x$ is invertible if and only if $x|_E$ is invertible for all simple subcoalgebras $E \subseteq C$.)

Proof. (a) The $C_n$'s form a filtration of $C$. Thus if $x \equiv 0$ on $C_0$, then $x^{n+1} \equiv 0$ on $C_n$ for $n > 0$ by induction, and therefore $\Sigma_{n=0}^{\infty} \alpha_n x^n$ is meaningful for all $\alpha_0, \alpha_1, \ldots \in k$. That $X \mapsto x$ extends to an algebra map is easy to check. If $x \equiv e$ on $C_0$, we have just shown that $X \mapsto e - x$ determines an algebra map, so $x = e - (e - x)$ is invertible since $1 - X \in k[[X]]$ is.

(b) If $x$ is left invertible, then $x|_D$ is for any subcoalgebra $D \subseteq C$. On the other hand, if $x|_E$ is left invertible for all simple subcoalgebras $E \subseteq C$, then $x|_{C_0}$ has a left inverse $f \in \text{Hom}(C_0, A)$ since $C_0$ is a direct sum of simples. Let $F: C \to A$ be a linear extension of $f$. Then $F \cdot x \equiv e$ on $C_0$ which means $F \cdot x$ is invertible by (a), hence $x$ is left invertible. The rest easily follows. Q.E.D.

Proposition 1. Let $C$ be a coalgebra and $A$ be an algebra over a field $k$, and let $f \in \text{Hom}(C, A)$.

(a) Let $L(f) \subseteq C$ (resp. $R(f) \subseteq C$) be the sum of all subcoalgebras $D \subseteq C$ such that $f|_D$ is left (resp. right) invertible. Then $f|_{L(f)}$ is left invertible and $f|_{R(f)}$ is right invertible.

(b) $C(f) = L(f) \cap R(f)$ is the sum of all subcoalgebras $D \subseteq C$ such that $f|_D$ is invertible, and $f|_{C(f)}$ is invertible.

(c) $L(f)$ and $R(f)$ (hence $C(f)$) are closed under wedging.
(d) If $\Omega$ is a field extension of $k$ then $L(f) \otimes_k \Omega \subseteq L(f \otimes_I \Omega)$ and $R(f) \otimes_k \Omega \subseteq R(f \otimes_I \Omega)$ (hence $C(f) \otimes_k \Omega \subseteq C(f \otimes_I \Omega)$).

(e) If $A$ is a bialgebra and $f$ is a coalgebra homomorphism, then $f|_{C(f)}^{-1}$ is a coalgebra antihomomorphism.

Proof. (a) follows by Lemmas 2(b) and 1(a). (b) follows directly from (a). That $L(f)$ and $R(f)$ are closed under wedging, or equivalently, $L(f) \wedge L(f) \subseteq L(f)$ and $R(f) \wedge R(f) \subseteq R(f)$, follows by Lemmas 2(b) and 1(b). (d) is straightforward. The proof of [HS, 1.5.2.(d)] generalizes to a proof of (e). Q.E.D.

Let $A$ be a bialgebra over $k$ and let $L(I), R(I)$ and $A(I)$ denote the subcoalgebras of $C = A$ described in Proposition 1 for the identity map $I$ of $A$. One should note that $G(A(I))$ consists of all invertible $g \in G(A)$ and is therefore a group. (For a coalgebra $C$ over $k$ recall that $g \in C$ is grouplike if $g \neq 0$ and $\Delta g = g \otimes g$, and that $G(C)$ denotes the set of grouplike elements of $C$.) By Proposition 1(e) $s = (I|_{A(I)})^{-1}$ is a coalgebra antihomomorphism. Observe that $s(g) = g^{-1}$ for $g \in G(A(I)).$

Suppose that $U$ and $V$ are vector spaces over $k$. For a field extension $\Omega$ of $k$: $(U \otimes_k \Omega) \otimes_k (V \otimes_k \Omega) \cong (U \otimes_k V) \otimes_k \Omega((u \otimes \alpha) \otimes (v \otimes \beta) \mapsto (u \otimes v) \otimes \alpha \beta)$ is an isomorphism of $\Omega$-spaces. The Galois group $G(\Omega \setminus k)$ acts on $U \otimes_k \Omega$ as automorphisms by the rule $\sigma \cdot (u \otimes \alpha) = u \otimes \sigma \alpha$. (Thus $G(\Omega \setminus k)$ acts as $k$-algebra automorphisms if $U$ is an algebra.)

Lemma 3. Let $C$ be a coalgebra over a field $k$ and suppose $\Omega$ is a field extension of $k$. Then $G(C \otimes_k \Omega)$ is a $G(\Omega \setminus k)$-module, which is cyclic if $C$ is cocommutative and simple and $\Omega$ is an algebraic closure of $k$.

Proof. If $g = \sum c_i \otimes \alpha_i \in C \otimes_k \Omega$, then $\Delta g = g \otimes g$ if and only if $\sum \Delta c_i \otimes \alpha_i = \sum_{ij} c_i \otimes c_j \otimes \alpha_i \alpha_j$. From this it follows that $G(C \otimes_k \Omega)$ is a $G(\Omega \setminus k)$-module. Now assume $C$ is cocommutative and simple, and $\Omega$ is an algebraic closure of $k$. The isomorphism $C \otimes_k \Omega \cong \text{Hom}_k(C^*, \Omega)$ $(c \otimes \alpha(c^*) = c^*(c)\alpha)$ restricts to an identification of $G(\Omega \setminus k)$-modules $G(C \otimes_k \Omega) \cong \text{Alg}_k(C^*, \Omega)$. Since $C^*$ is a finite-dimensional field extension of $k$ and $\Omega$ is an algebraic closure of $k$, given $\tau, \tau' \in \text{Alg}_k(C^*, \Omega)$ there exists a $\sigma \in G(\Omega \setminus k)$ such that $\tau' = \sigma \circ \tau$, i.e. $\text{Alg}_k(C^*, \Omega)$ is cyclic. Q.E.D.

Let $V$ be a left $C$-module with basis $v_1, \ldots, v_n$. Define $e_{ij} \in C$ $(1 < i, j < n)$ by $\omega(v_i) = \sum e_{ij} \otimes v_j$. Then $\Delta e_{ij} = \sum_k e_{ik} \otimes e_{kj}$ and $\varepsilon(e_{ij}) = \delta_{ij}$ follow from the comodule axioms. If $C$ is a commutative bialgebra then the determinant $d = \text{det}(e_{ij})$ of $(e_{ij}) \in M(n, C)$ is grouplike and does not depend on the choice of basis. Here we set $d_F = d$.

Let $C(n, k)$ be the coalgebra over $k$ with basis of symbols $e_{ij} (1 < i, j < n)$ with structure defined as above. Let $V \subseteq C(n, k)$ have basis $e_{11}, \ldots, e_{nn}$ and let $S(C(n, k))$ be the free commutative bialgebra on $C(n, k)$. Then $\Omega_{k}(k) = S(C(n, k))[d_F^{-1}]$ is a Hopf algebra since it represents the affine group scheme $\text{GL}_n(\cdot)$ over $k$. 

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2. The main results. Here we examine the role of grouplikes in certain bialgebras.

Proposition 2. Let $A$ be a bialgebra over a field $k$ which is commutative or has cocommutative coradical. Then $A_{(I)}$ is a Hopf algebra.

Proof. Assume $A$ is commutative. We will use Nichols’ result [N] that a bialgebra quotient of a commutative Hopf algebra is a Hopf algebra. First let $V \subseteq A_{(I)}$ be a simple subcoalgebra and choose $e_{ij} \in A$ ($1 < i, j < n$) for $V$ as indicated above. Since $V$ is a coalgebra, the $e_{ij}$’s span $V$. From the equations $\Sigma_k e_{ik}s(e_{kj}) = \delta_{ij}1$ we conclude that $\{e_{ij}\} \in M(n, A)$ is invertible, hence $d_V$ is invertible. Clearly $B_V = (V)[d_V^{-1}]$ is a bialgebra quotient of $\otimes_n(k)$, so $B_V$ is a Hopf algebra. In particular $s(V) \subseteq A_{(I)}$. For simple subcoalgebras $E', E'' \subseteq A_{(I)}$ we apply Nichols’ result again to multiplication $B_{E'} \otimes B_{E''} \rightarrow B_E B_{E''}$ to deduce $E'E'' \subseteq A_{(I)}$. Thus $A_{(I)}$ is a Hopf algebra by Lemmas 2(b) and 1(c).

Now assume $A_0$ is cocommutative and $\beta$ is an algebraic closure of $k$. Then $A \otimes_k \beta$ is pointed. $D = A(I) + s(A(I)^\beta)$ is a subcoalgebra of $A$ by Proposition 1(e). Note $G(s(A(I)) \otimes_k \beta) = G(s \otimes I(A(I) \otimes_k \beta)) = G(A(I) \otimes_k \beta)^{-1}$ by Lemma 1(d), so $G(D \otimes_k \beta) = G(D \otimes_k \beta)^{-1}$ by part (a) of the same lemma. Thus by Lemma 1(c) the grouplikes of $(D) \otimes_k \beta = (D \otimes_k \beta)^{-1}$ form a group. Therefore $(D) \otimes_k \beta$ is a Hopf algebra by Lemma 2(b), and this means $(D)$ is a Hopf algebra also. By definition $A_{(I)} = (D)$. Q.E.D.

Using Lemma 3 we have as a corollary to the proof:

Corollary 1. Let $A$ be a bialgebra over a field $k$, and suppose $C \subseteq A$ is a simple subcoalgebra.

(a) If $A$ is commutative, then $C \subseteq A_{(I)}$ if and only if $d_C$ is invertible in $A$.

(b) If $A_0$ is cocommutative and $\Omega$ is an algebraic closure of $k$, then $C \subseteq A_{(I)}$ if and only if some $g \in G(C \otimes_k \Omega)$ is invertible in $A \otimes_k \Omega$.

$A_{(I)}$ need not be a Hopf algebra in general.

Example 1. Let $C = C(n, k) \oplus C(n, k)$ and $T(C)$ be the free bialgebra on $C$. The ideal $I \subseteq T(C)$ generated by the relations described in $M(n, T(C))$ by $(e_{ij})(e_{ij})' = I = (e_{ij})(e_{ij})'$ is a bi-ideal. By the method of §1 of [B] one can show that $C \subseteq A \equiv T(C)/I$ (so $C(n, k) \subseteq A_{(I)}$) and that $(e_{ij}) \in M(n, A)$ is not invertible (so $s(C(n, k)) = C(n, k) \not\subseteq A_{(I)}$) for $n > 2$.

Proposition 3. Let $A$ be a bialgebra over a field $k$ and suppose $C, C' \subseteq A$ are nonzero subcoalgebras such that $CC' \subseteq A_{(I)} \supseteq C'$. Then $C, C' \subseteq A_{(I)}$.

Proof. Assume $CC' \subseteq A_{(I)}$ and choose $a' \in C'$ such that $e(a') = 1$. Then $t \in \text{Hom}(C, A)$ defined by $t(c) = \Sigma a'(c)\Sigma a'(c') = I = (e_{ij})(e_{ij})'$ is a right inverse for $I|_C$, so $C \subseteq R_{(I)}$. Likewise $C' \subseteq L_{(I)}$. Q.E.D.

Let $A$ be a bialgebra with left antipode $s$ (a left inverse of $I \in \text{Hom}(A, A)$). Such an $A$ is a left Hopf algebra. Suppose $V \subseteq A$ is a left Hopf submodule (i.e. $\Delta V \subseteq A \otimes V$ and $AV \subseteq V$). For $v \in V$ the calculation $e(v)1 = \Sigma s(v(1))v(2) \in AV = V$ shows $V = (0)$ or $V = A$, so $A$ is a simple left $A$-Hopf module.
Theorem 1. Let $A$ be a bialgebra over a field $k$ which is commutative or has cocommutative coradical. Then the following are equivalent.

(a) $A$ is a Hopf algebra.
(b) $A$ is a simple left $A$-Hopf module.
(c) If $C \subseteq A$ is a simple subcoalgebra then $AC = A$.

Proof. We need only show (c) $\Rightarrow$ (a). Assume (c) holds and let $C \subseteq A$ be simple. By Lemma 2(b) we need only show $C \subseteq A_{(1)}$. If $A$ is commutative, $d_C$ is invertible, so $C \subseteq A_{(1)}$ by Corollary 1(a). Suppose $A_0$ is cocommutative and $\Omega$ is an algebraic closure of $k$. Then $AC = A$ means $1 \in C'C$ for some simple $C'$ by Lemma 1(c), so $1 = g'g$ where $g' \in G(C' \otimes_k \Omega)$ and $g \in G(C \otimes_k \Omega)$ by the same result. Now replacing $C$ by $C'$ we see $1 = h''h'$ for some $h' \in G(C' \otimes \Omega)$. Thus the calculation $1 = (\sigma \cdot h'')(\sigma \cdot h')$ shows that $g'$ is invertible, so $g$ is also, and $C \subseteq A_{(1)}$ by Corollary 1(b).

A bialgebra which is a simple left Hopf module may not be a Hopf algebra (there exist left Hopf algebras which are not Hopf algebras [GNT]).

As a consequence of Lemma 2(b) and Corollary 1(a) a commutative bialgebra is a Hopf algebra if and only if all grouplikes are invertible [T, Corollary 69]. Generally this is not the case.

Example 2. Let $C$ be a coalgebra over a field $k$ and let $T(C)$ be the free bialgebra on $C$. Then $T(C)^n = C \otimes \cdots \otimes C$ ($n$-times) has the tensor product coalgebra structure ($n > 1$). From the general coalgebra fact that $G(C_1 \otimes \cdots \otimes C_n) = G(C_1) \times \cdots \times G(C_n)$ and Lemma 1(a) we conclude that $G(T(C)) = \{1\}$ if $G(C) = \emptyset$. For example let $k = \mathbb{R}$ and $C = C^*$. Then $T(C)$ is a cocommutative bialgebra, but is not a Hopf algebra, since $T(C) \otimes_{\mathbb{R}} C \simeq T(C \otimes_{\mathbb{R}} C)$ is the free monoid on $G(C \otimes_{\mathbb{R}} C)$.

Example 3. Let $C$ be a coalgebra which is also an algebra (possibly without unity), and suppose $\Delta$, $\varepsilon$ are multiplicative. The coalgebra structure of $C$ extends (uniquely) to a bialgebra structure on the algebra $A = k \cdot 1 + C$ obtained by adjoining a unity to $C$. $A$ is not a Hopf algebra, since $C$ is a sub-Hopf module, unless $C = (0)$. For finite-dimensional examples where $G(A) = \{1\}$ let $C = C(n, k)$ and $e_{ij}e_{kl} = \delta_{ij}e_{kl}$.

Our last result gives a module-theoretic characterization of commutative or cocommutative Hopf algebras.

Theorem 2. Let $A$ be a bialgebra over a field $k$ which is commutative or has cocommutative coradical. Then $A$ is a Hopf algebra if and only if all left $A$-Hopf modules are free (or $(0)$).

Proof. If $A$ is any Hopf algebra over a field, then all left Hopf modules are free (or $(0)$) by [S, Theorem 4.1.1]. Conversely, suppose $A$ satisfies this condition and let $C \subseteq A$ be a simple subcoalgebra. By Theorem 1 we must show $AC = A$, or $A/AC = (0)$. Since $AC$ and $A/AC$ are Hopf modules, $AC$ is free and $A/AC$ is free or $(0)$. Since ker $\varepsilon$ is a codimension 1 ideal, any two bases of a free $A$-module $M$
have the same cardinality \( r(M) \). If \( A/AC \) is free, then \( 1 = r(A) = r(AC) + r(A/AC) \), a contradiction, so \( A/AC = (0) \). \( \text{Q.E.D.} \)

“Free” cannot be replaced by “projective” in the preceding theorem since there are semisimple bialgebras which are not Hopf algebras. We close with a general construction.

**Example 4.** Let \( \mathcal{U} \) be an associative algebra (with unity) over \( k \) with an algebra map \( \delta: \mathcal{U} \to \mathcal{U} \otimes \mathcal{U} \) satisfying \( I \otimes \delta \circ \delta = \delta \otimes I \circ \delta \). The direct sum of algebras \( A = k \cdot d \oplus \mathcal{U} \) \((d^2 = d \neq 0)\) has a bialgebra structure determined by \( \Delta d = d \otimes d \) and \( \Delta a = a \otimes d + d \otimes a + \delta a \) for \( a \in \mathcal{U} \). \( A \) is a Hopf algebra if and only if \( \mathcal{U} = (0) \). Let \( \mathcal{U} \) be any semisimple bialgebra.

**References**


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