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TRACE AND THE REGULAR RING OF A FINITE AW*-ALGEBRA

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ABSTRACT. A finite AW*-algebra is of type I if and only if its maximal ring of quotients has a center-valued trace. In particular, a center-valued trace need not be extendible to the maximal (or classical) ring of quotients.

Let R be a ring with involution $x \mapsto x^*$ (that is, $x^{**} = x$, $(x + y)^* = x^* + y^*$ and $(xy)^* = y^*x^*$ for all x, y in R), and let Z be the center of R; we say that R *has a center-valued trace* if there exists a mapping $R \to Z$, denoted $x \mapsto x^\natural$, such that (i) $(x + y)^\natural = x^\natural + y^\natural$ for all x, y in R, (ii) $(xy)^\natural = (yx)^\natural$ for all x, y in R, (iii) $z^\natural = z$ for all $z \in Z$, and (iv) for each $x \in R$, $(x^*x)^\natural$ is a finite sum of elements of the form z^*z with $z \in Z$ (a "positivity" condition).

Let A be a finite AW*-algebra and let C be its regular ring as constructed, for example, in [**4**, Chapter 8]. {Thus C is a regular Baer $*$-ring [**4**, Theorem 1, p. 220 and Theorem 1, p. 235]. We remark that, by a theorem of J. E. Roos [**10**], C may be identified with the maximal ring of quotients (right or left) of A (cf. [**7**], [**9**]). Moreover, for every $x \in C$ one can write $x = ab^{-1}$ with $a = x(1 + x^*x)^{-1}$ and $b = (1 + x^*x)^{-1}$, and one has $a^*a \leqslant 1$, $0 \leqslant b \leqslant 1$ (cf. [**4**, Exercise 3, p. 242]); it follows that C is a classical ring of quotients of A [**8**, p. 108], a remark valid with A replaced by any finite Baer $*$-ring satisfying the axioms $1°$–$6°$ of [**4**, pp. 248–249]. In the foregoing remarks, statements about right quotients and left quotients are equally valid, since the involution of A extends to C.}

It is not known if, in general, A possesses a center-valued trace; it is known that a trace exists if A is of type I (cf. [**2**, proof of Theorem 5, p. 178]) or if A is a (finite) von Neumann algebra [**6**, Theorem 1, p. 288]. In the following theorem, the question of existence of trace for A is not begged; the theorem shows that even if A possesses a trace, C need not.

THEOREM 1. *Let A be a finite* AW*-*algebra, C its regular ring. In order that C admit a center-valued trace, it is necessary and sufficient that A be of type* I.

PROOF. The proof that C possesses a center-valued trace when A is of type I is given in [**2**, Theorem 5]. In general, A is the sum of a type I algebra and a type II algebra (cf. [**4**, Theorem 2, p. 94]); assuming A to be of type II, the proof will be completed by showing that C does not possess a center-valued trace. Let (g_n) be a sequence of pairwise orthogonal projections in A such that $\sup g_n = 1$ and $D(g_n) = 2^{-n}1$ for all n ($n = 1, 2, 3, \cdots$), D being the center-valued dimension function

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of A [cf. **4**, Theorem 1, p. 181 and Proposition 15, p. 159]. Write $x_n = \sum_{k=1}^{n} 2^k g_k$, $e_n = \sum_{k=1}^{n} g_k$; if $m < n$ then $x_n e_m = x_m = e_m x_n$, thus there exists $x \in C$ such that $xe_n = x_n$ for all n [**4**, Proposition 1, p. 219]. Then [**4**, Proposition 6, p. 242] one has $x \geqslant 0$ and $x_n = x^{1/2} e_n x^{1/2} \leqslant x^{1/2} 1 x^{1/2} = x$, thus $0 \leqslant x_n \leqslant x$ for all n. Assume to the contrary that C admits a center-valued trace $\natural$. It follows from uniqueness of dimension that $e^{\natural} = D(e)$ for all projections e [**4**, Theorem 1, p. 181]; therefore $(x_n)^{\natural} = \sum_{k=1}^{n} 2^k D(g_k) = n1$, thus $0 \leqslant n1 = (x_n)^{\natural} \leqslant x^{\natural}$ for all n. Then $0 \leqslant (x^{\natural})^{-1} \leqslant (n1)^{-1}$ for all n [3, Proposition 8.12], thus $(x^{\natural})^{-1} \in A$ and $\|(x^{\natural})^{-1}\| \leqslant 1/n$ for all n, which yields the absurdity $(x^{\natural})^{-1} = 0$. □

It is curious that although C does not in general possess a trace, it exhibits the following trace-like behavior: the equation $x^*x - xx^* = 1$ has no solution in C [**5**, Lemma, p. 619]. Also, Theorem 1 thwarts any prospect of proving a theorem of Fuglede type in C by means of a trace argument (cf. [**2**, Theorems 4 and 5], [**5**, Theorem 6]).

For A a finite Baer $*$-ring, it is not clear what "trace" should mean. The dimension function D takes its values in the space of continuous complex-valued functions $\mathcal{C}(\mathfrak{X})$, where $\mathfrak{X}$ is the Stone representation space (i.e., the spectrum) of the complete Boolean algebra of central projections of A [**4**, p. 153]. Since not every element of A need be bounded in the sense of [**4**, Definition 1, p. 243], and since a trace function should in some sense extend the dimension function, a good candidate for the value-space of a trace is the regular ring $\hat{\mathcal{C}}(\mathfrak{X})$ of the commutative AW*-algebra $\mathcal{C}(\mathfrak{X})$. Let us say that A *has a spectral trace* if there exists a mapping $A \to \hat{\mathcal{C}}(\mathfrak{X})$, denoted $x \mapsto x^{\natural}$, such that (i) $(x + y)^{\natural} = x^{\natural} + y^{\natural}$, (ii) $(xy)^{\natural} = (yx)^{\natural}$, (iii) $h^{\natural} = h$ for all central projections h, and (iv) $(x^*x)^{\natural} \geqslant 0$ for all $x \in A$. {When A is a finite AW*-algebra, the concept of spectral trace coincides with that of center-valued trace defined earlier; for, in this case, the center Z of A may be identified with $\mathcal{C}(\mathfrak{X})$, and the center of C with $\hat{\mathcal{C}}(\mathfrak{X})$ [**1**, Theorem 9.2].} For A a finite Baer $*$-ring of type I, the construction of trace in Theorem 1 breaks down (basically because an abelian ring need not be commutative), and, as the following theorem shows, the bad news persists for rings of type II (so to speak, $\hat{\mathcal{C}}(\mathfrak{X})$ is no better a value space for trace, than is the center of C).

THEOREM 2. *If A is a finite Baer $*$-ring of type* II, *satisfying the axioms* 1°–5° *of* [**4**, *p.* 248], *then the regular ring C of A does not admit a spectral trace.*

PROOF. The proof proceeds as in Theorem 1, up through the point that $0 \leqslant (x^{\natural})^{-1} \leqslant (n1)^{-1}$ for all n. In particular, $0 \leqslant (x^{\natural})^{-1} \leqslant 1$, therefore there exists a projection e such that $D(e) = (x^{\natural})^{-1}$ [**4**, Theorem 3, p. 182]. Let f be a simple projection such that $f \leqslant e$ [**4**, Proposition 16, p. 159], let h be the central cover of f, and let r be the integer such that $D(f) = (1/r)h$; then $0 \leqslant (1/r)h = D(f) \leqslant D(e) = (x^{\natural})^{-1} \leqslant (n1)^{-1}$ for all n, and for $n = 2r$ this yields $2h \leqslant 1$, $h \leqslant 1 - h$, whence $h = 0$, a contradiction. □

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