RENORMING THE BANACH SPACE \( c_0 \)

ROBERT C. JAMES

ABSTRACT. Let \( \Phi \) be a subset of the unit sphere of \( l_1 \) and let \( X \) be \( c_0 \) renormed by using \( \Phi \) and letting \( x = y \) if \( \| x - y \| = 0 \). Two conditions are given, which together imply \( X \) is "almost isometric" to a subspace of \( c_0 \). One condition is satisfied if \( \Phi \) is the unit sphere of a linear subset of \( l_1 \). Both conditions are satisfied if \( X \) is a quotient \( c_0/W \) and \( \Phi \) is the subset of the unit sphere whose members are zero on \( W \).

A Banach space \( X \) is said to be almost isometric to a subspace of \( c_0 \) if for each positive \( \varepsilon \) there is an isomorphism \( T \) of \( X \) onto a subspace of \( c_0 \) for which \( \| T \| = 1 \) and \( \| T^{-1} \| < 1 + \varepsilon \). It will be proved that, if \( \Phi \) is a \( Q \)-subset of \( l_1 \) as defined below, then \( X \) is almost isometric to a subspace of \( c_0 \) if \( X \) is obtained from \( c_0 \) by using \( \Phi \) to determine a new norm \( \| \cdot \| \) and then letting \( x = y \) if \( \| x - y \| = 0 \).

For each positive integer \( n \), let \( P_n \) denote the natural projection onto the span of the first \( n \) members of the usual basis of \( l_1 \). If \( \Phi \) is a subset of the unit sphere of \( l_1 \), let \( \Phi_n \) denote the set of those members of \( \Phi \) whose first \( n - 1 \) components are 0. A subspace of a Banach space will be a linear subset, not necessarily closed.

DEFINITION. A \( Q \)-subset of \( l_1 \) is a subset \( \Phi \) of the unit sphere of \( l_1 \) that has the properties:

A) If \( \theta + h_i \in \Phi \) for each \( i \), \( \theta \neq 0 \), \( \| h_i \| \) is bounded away from 0, and \( \lim_{n \to \infty} \| P_n h_i \| = 0 \) for each \( n \), then \( \theta/\| \theta \| \in \Phi \) and \( \text{dist}(h_i/\| h_i \|, \Phi) \to 0 \).

B) For each \( n \) and \( \tau > 0 \), there is a \( \sigma > 0 \) and a finite subset \( H \) of \( \Phi_n \) such that, if \( \varphi \in \Phi_n \) and \( |\varphi(n)| < \sigma \), then there is a \( g \) in \( H \cup \Phi_{n+1} \) such that \( \| \varphi - g \| < \tau \).

It is very easy to prove and it has long been known that every separable Banach space is a quotient space of \( l_1 \) [4, p. 108]. The behavior of \( c_0 \) is quite different. In fact, Alspach showed recently that each quotient space of \( c_0 \) is almost isometric to a subspace of \( c_0 \) [1]. It had been known for some time that each quotient space of \( c_0 \) is isomorphic to a subspace of \( c_0 \) ([3, p. 53] or [4, p. 107]). Rather than studying quotient spaces of \( c_0 \), the following theorem and its proof demonstrate that it is more natural, much easier, and simpler to study renormings of \( c_0 \) by subsets of the unit sphere of \( l_1 \).

It is easy to show that, given any separable Banach space \( X \), there is a subset \( \Phi \) of the unit sphere of \( l_1 \) which, when used to renorm \( c_0 \), produces a new space that contains a subset isometric to a dense linear subset of \( X \). However, we will prove...
that \(Q\)-subsets of \(l_1\) can only generate spaces which are almost isometric to subspaces of \(c_0\).

For example, let \(\Phi = \{ \frac{1}{2}e_n - \frac{1}{2}e_{n+1}; n > 1 \}\), where \(\{e_n\}\) is the natural basis of \(l_1\).
Then \(\Phi\) is a \(Q\)-subset of \(l_1\). If \(X\) is the space obtained by using \(\Phi\) to renorm \(c_0\), then the identity map \(I\) of \(c_0\) onto \(X\) has only 0 in its kernel. Also,

\[
\|\|(1, 1 - 1/k, 1 - 2/k, \ldots, 1 - (k - 1)/k, 0, 0, \ldots)\|\| = 1/(2k),
\]

so \(I^{-1}\) is not continuous. Thus \(X\) is not complete and is not a quotient of \(c_0\), but \(\{x_n\} \rightarrow \{\frac{1}{2}(x_1 - x_2), \frac{1}{2}(x_2 - x_3), \ldots\}\) defines an isometry of \(X\) onto a subspace of \(c_0\).

Now let \(\Phi = \{ \frac{1}{2}e_1 - \frac{1}{2}e_n; n > 1 \}\). Then \(\Phi\) has property (B) and does not have property (A). The space \(X\) obtained by renorming \(c_0\) is isometric to \(c\), with \(\{x_i; i > 1\} \leftrightarrow \{\frac{1}{2}(x_1 - x_i); i > 1\}\).

Let us say that \(\Phi\) is prelinear if \(\Phi\) is the unit sphere of a subspace of \(l_1\). If \(\Phi\) is prelinear, then \(\Phi\) has property (B). In fact, if \(\Phi_{n+1} \neq \Phi_n\), there is an \(f\) with \(f/\|f\| \in \Phi_n, f(n) > 0\), and \(\|f\| < \frac{1}{2}\tau/(1 + \tau)\). We then let \(H = \emptyset\) and \(\sigma = f(n)\). If \(\varphi(n) < \sigma\), let

\[
g = \frac{\varphi - \varphi(n)/\sigma}{\|\varphi - \varphi(n)/\sigma\|}.
\]

However, \(\Phi\) can be prelinear and not have property (A). To see this, let \(\Phi\) be the subset of the unit sphere of \(l_1\) for which \(\Sigma_i^\infty f_i = 0\) if \(\{f_i\} \in \Phi\). Then \(\frac{1}{2}e_1 - \frac{1}{2}e_n \in \Phi\) for all \(n > 1\), but \(e_1 \notin \Phi\).

A quotient \(c_0/W\) can be regarded as obtained by renorming \(c_0\) using as \(\Phi\) the intersection of \(W^\perp\) and the unit sphere of \(l_1\). Such a \(\Phi\) clearly has property (A) and is a \(Q\)-subset of \(l_1\). Thus it is a corollary of the following theorem that quotient spaces of \(c_0\) are almost isometric to subspaces of \(c_0\). It should be noted that if a \(Q\)-subset \(\Phi\) of \(l_1\) is prelinear, then it follows from (A) that \(\text{cl[lin(\Phi)]}\) is bw*-closed, which implies \(\text{cl[lin(\Phi)]}\) is w*-closed [2, Theorem 5, p. 50] and therefore that there is a subspace \(W\) of \(c_0\) for which \(W^\perp = \text{cl[lin(\Phi)]}\).

**Theorem.** Let \(\Phi\) be a \(Q\)-subset of \(l_1\). Then \(X\) is almost isometric to a subspace of \(c_0\) if \(X\) is obtained by renorming \(c_0\) with

\[
\|x\| = \sup\{|(\varphi, x)|; \varphi \in \Phi\}
\]

and \(x = y\) if \(\|x - y\| = 0\).

**Proof.** It is sufficient to show that, for each \(\varepsilon > 0\), there exists a sequence \(\{\varphi_i; i > 1\}\) in \(\Phi\) such that \(w^*\)-lim \(\varphi_i = 0\) and

\[
\sup\{|(\varphi_i, x)|; i > 1\} > (1 + \varepsilon)^{-1}\|x\|\quad \text{if} \quad x \in X,
\]

since then \(\|T\| > 1\) and \(\|T^{-1}\| < 1 + \varepsilon\) if \(T\) is defined by \((Tx)(i) = \varphi_i(x)\) for \(i > 1\). Suppose \(\Delta_n\) is a positive number and \(F_n\) is a finite subset of \(\Phi\) such that, if \(x \in X\), then

\[
\sup\{|(f, x)|; f \in F_n \cup \Phi_n\} > (1 - \Delta_n)\|x\|.
\]

If \(n = 1\), this is satisfied with \(F_1 = \emptyset\) and \(\Delta_1\) any positive number. We will show that if \(\Delta_{n+1} > \Delta_n\), then there is a finite subset \(G_n\) of \(\Phi_n\) such that, if \(x \in X\), then
\[ \sup \{|(f, x)|; f \in F_n \cup G_n \cup \Phi_{n+1}^r \} > (1 - \Delta_n - \tau) ||x||. \]  

Once this has been done, the existence of \( \{\varphi_i\} \) follows by choosing \( \Delta_n \) uniformly less than \( 1 - (1 + \epsilon)^{-1} \), and then letting \( F_{n+1} = F_n \cup G_n \) and \( \{\varphi_i\} = \cup F_n \), arranged in an order so that each member of \( F_n \) precedes each member of \( G_n = F_{n+1} - F_n \).

Suppose (i) is satisfied, but (ii) is not satisfied, whatever finite subset \( G_n \) of \( \Phi_n \) is used. This implies that, if \( \tau = \frac{1}{2}(\Delta_{n+1} - \Delta_n) \) and \( \sigma \) is as described in (B), then for each \( k > 1 \) there is an \( x_k \in X \) such that \( ||x_k|| = 1 \) and

\[ \sup \{|(f, x)|; f \in F_n \cup G^r \cup \Phi_{n+1}^r \} < (1 - \Delta_n - \tau) ||x|| \]

is satisfied if \( x = x_k \), where \( G^r \) is the set of all \( \varphi \in \Phi_n \) with \( ||\varphi - P_{n+k}\varphi|| < \frac{1}{3} \tau \), and \( \Phi_{n+1}^r \) is the set of all \( \varphi \in \Phi_n \) with \( ||\varphi(n)|| < \sigma \). To see this, observe first that, if \( G \) is a finite subset of \( G^r \) for which \( P_{n+k}G \) is \( \frac{1}{3} \tau \)-dense in \( P_{n+k}G^r \), then \( G \) is \( \tau \)-dense in \( G^r \). Then the left member of (iii) is not decreased by more than \( \tau ||x|| \) if \( G^r \) is replaced by \( G \). Now observe that, for \( H \) as given by (B), \( \Phi_{n+1}^r \) can be replaced by \( H \cup \Phi_{n+1}^r \) without decreasing the left member of (iii) by more than \( \tau ||x|| \) for any \( x \). It now follows that \( x_k \) exists, since otherwise (ii) would be satisfied with \( G_n = G \cup H \). This \( x_k \) has the property

if \( \varphi \in \Phi_n \) and \( |(\varphi, x_k)| > 1 - \Delta_n - \tau \), then \( \varphi \not\in F_n \), \( |\varphi(n)| > \sigma \),

and \( ||\varphi - P_{n+k}\varphi|| > \frac{1}{3} \tau \).

Because of (i) and the fact that \( x_k \) satisfies (iii), there is \( g_k \in \Phi_n \) for which \( |(g_k, x_k)| > 1 - \Delta_n \). It then follows from (1) that

\[ |g_k(n)| > \sigma \quad \text{and} \quad ||g_k - P_{n+k}g_k|| > \frac{1}{3} \tau. \]

Now choose a subsequence of \( \{g_k\} \) which converges component-wise. Let \( \theta \) be this component-wise limit, denote the subsequence by \( \{\theta + h_k\} \), and let the corresponding members of \( \{x_k\} \) be \( \{\xi_k\} \). Then (1) is satisfied with \( x_k \) replaced by \( \xi_k \). Also,

\[ |(\theta + h_k, \xi_k)| > 1 - \Delta_n. \]

It follows from (2) and the choice of \( \{\theta + h_k\} \) that, for each positive integer \( \lambda \),

\[ |\theta(n)| > \sigma, \quad \lim_{k \to \infty} ||h_k|| > \frac{1}{3} \tau, \quad \text{and} \quad \lim_{k \to \infty} ||P_\lambda h_k|| = 0. \]

It follows from (A) that \( \theta/||\theta|| \in \Phi_n \) and \( \text{dist}(h_k/||h_k||, \Phi_n) \to 0 \). Choose \( \lambda > n \) so that \( ||\theta - P_\lambda \theta|| < \frac{1}{6} \tau ||\theta|| \). Then it follows from (1) that

\[ |(\theta, \xi_k)/||\theta||| < 1 - \Delta_n - \tau \quad \text{if} \quad n + k > \lambda. \]

Now choose \( k \) so that \( n + k > \lambda \), \( |h_k(n)| < \frac{1}{2} \sigma ||h_k|| \), \( ||P_\lambda h_k|| < \frac{1}{6} \tau \), and \( \text{dist}(h_k/||h_k||, \Phi_n) \) is small enough that it follows from \( |h_k(n)| < \frac{1}{2} \sigma ||h_k|| \) and (1) that

\[ |(h_k, \xi_k)/||h_k||| < 1 - \Delta_n - \tau + \frac{1}{2} \tau/||h_k||. \]

Recall that \( ||\theta + h_k|| = 1 \) and observe that the conditions on \( \lambda \) and \( k \) imply

\[ ||\theta|| + ||h_k|| < ||\theta + h_k|| + 2||\theta - P_\lambda \theta|| + 2||P_\lambda h_k|| < 1 + \frac{1}{2} \tau. \]
Now use this and (5) and (6) to obtain a contradiction of (3):

\[
|\langle \theta + h_k, \xi_k \rangle| < \|\theta\| + \|h_k\| - (\|\theta\| + \|h_k\|)(\Delta_n + \tau) + \frac{1}{2}\tau
\]

\[
< 1 + \frac{1}{2}\tau - (\Delta_n + \tau) + \frac{1}{2}\tau = 1 - \Delta_n.
\]

REFERENCES


DEPARTMENT OF MATHEMATICS, CLAREMONT GRADUATE SCHOOL, CLAREMONT, CALIFORNIA 91711