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RENORMING THE BANACH SPACE c_0

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ABSTRACT. Let Φ be a subset of the unit sphere of l_1 and let X be c_0, renormed by using Φ and letting $x = y$ if $|||x - y||| = 0$. Two conditions are given, which together imply X is "almost isometric" to a subspace of c_0. One condition is satisfied if Φ is the unit sphere of a linear subset of l_1. Both conditions are satisfied if X is a quotient c_0/W and Φ is the subset of the unit sphere whose members are zero on W.

A Banach space X is said to be *almost isometric to a subspace of* c_0 if for each positive ε there is an isomorphism T of X onto a subspace of c_0 for which $\|T\| = 1$ and $\|T^{-1}\| < 1 + \varepsilon$. It will be proved that, if Φ is a Q-subset of l_1 as defined below, then X is almost isometric to a subspace of c_0 if X is obtained from c_0 by using Φ to determine a new norm $|||\ |||$ and then letting $x = y$ if $|||x - y||| = 0$.

For each positive integer n, let P_n denote the natural projection onto the span of the first n members of the usual basis of l_1. If Φ is a subset of the unit sphere of l_1, let Φ_n denote the set of those members of Φ whose first $n - 1$ components are 0. A *subspace* of a Banach space will be a linear subset, not necessarily closed.

DEFINITION. A *Q-subset* of l_1 is a subset Φ of the unit sphere of l_1 that has the properties:

(A) If $\theta + h_i \in \Phi$ for each i, $\theta \neq 0$, $\|h_i\|$ is bounded away from 0, and $\lim_{i\to\infty}\|P_n h_i\| = 0$ for each n, then $\theta/\|\theta\| \in \Phi$ and $\operatorname{dist}(h_i/\|h_i\|, \Phi) \to 0$.

(B) For each n and $\tau > 0$, there is a $\sigma > 0$ and a finite subset H of Φ_n such that, if $\varphi \in \Phi_n$ and $|\varphi(n)| < \sigma$, then there is a g in $H \cup \Phi_{n+1}$ such that $\|\varphi - g\| < \tau$.

It is very easy to prove and it has long been known that every separable Banach space is a quotient space of l_1 [4, p. 108]. The behavior of c_0 is quite different. In fact, Alspach showed recently that each quotient space of c_0 is almost isometric to a subspace of c_0 [1]. It had been known for some time that each quotient space of c_0 is isomorphic to a subspace of c_0 ([3, p. 53] or [4, p. 107]). Rather than studying quotient spaces of c_0, the following theorem and its proof demonstrate that it is more natural, much easier, and simpler to study renormings of c_0 by subsets of the unit sphere of l_1.

It is easy to show that, given any separable Banach space X, there is a subset Φ of the unit sphere of l_1 which, when used to renorm c_0, produces a new space that contains a subset isometric to a dense linear subset of X. However, we will prove

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that Q-subsets of l_1 can only generate spaces which are almost isometric to subspaces of c_0.

For example, let $\Phi = \{\frac{1}{2}e_n - \frac{1}{2}e_{n+1}: n \geqslant 1\}$, where $\{e_n\}$ is the natural basis of l_1. Then Φ is a Q-subset of l_1. If X is the space obtained by using Φ to renorm c_0, then the identity map I of c_0 onto X has only 0 in its kernel. Also,

$$|||(1, 1 - 1/k, 1 - 2/k, \ldots, 1 - (k-1)/k, 0, 0, \ldots)||| = 1/(2k),$$

so I^{-1} is not continuous. Thus X is not complete and is not a quotient of c_0, but $\{x_n\} \to \{\frac{1}{2}(x_1 - x_2), \frac{1}{2}(x_2 - x_3), \ldots\}$ defines an isometry of X onto a subspace of c_0.

Now let $\Phi = \{\frac{1}{2}e_1 - \frac{1}{2}e_n: n > 1\}$. Then Φ has property (B) and does not have property (A). The space X obtained by renorming c_0 is isometric to c, with $\{x_i: i \geqslant 1\} \leftrightarrow \{\frac{1}{2}(x_1 - x_i): i > 1\}$.

Let us say that Φ is *prelinear* if Φ is the unit sphere of a subspace of l_1. If ϕ is prelinear, then Φ has property (B). In fact, if $\Phi_{n+1} \neq \Phi_n$, there is an f with $f/\|f\| \in \Phi_n$, $f(n) > 0$, and $\|f\| < \frac{1}{2}\tau/(1+\tau)$. We then let $H = \varnothing$ and $\sigma = f(n)$. If $|\varphi(n)| \leqslant \sigma$, let

$$g = \frac{\varphi - \varphi(n)f/\sigma}{\|\varphi - \varphi(n)f/\sigma\|}.$$

However, Φ can be prelinear and not have property (A). To see this, let Φ be the subset of the unit sphere of l_1 for which $\sum_1^\infty f_i = 0$ if $\{f_i\} \in \Phi$. Then $\frac{1}{2}e_1 - \frac{1}{2}e_n \in \Phi$ for all $n > 1$, but $e_1 \notin \Phi$.

A quotient c_0/W can be regarded as obtained by renorming c_0, using as Φ the intersection of $W^\perp$ and the unit sphere of l_1. Such a Φ clearly has property (A) and is a Q-subset of l_1. Thus it is a corollary of the following theorem that quotient spaces of c_0 are almost isometric to subspaces of c_0. It should be noted that if a Q-subset Φ of l_1 is prelinear, then it follows from (A) that cl[lin(Φ)] is bw^*-closed, which implies cl[lin(Φ)] is w^*-closed [2, Theorem 5, p. 50] and therefore that there is a subspace W of c_0 for which $W^\perp$ is cl[lin(Φ)].

THEOREM. *Let Φ be a Q-subset of l_1. Then X is almost isometric to a subspace of c_0 if X is obtained by renorming c_0, with*

$$|||x||| = \sup\{|(\varphi, x)|: \varphi \in \Phi\} \quad \textit{and} \quad x = y \textit{ if } |||x - y||| = 0.$$

PROOF. It is sufficient to show that, for each $\varepsilon > 0$, there exists a sequence $\{\varphi_i: i \geqslant 1\}$ in Φ such that w^*-lim $\varphi_i = 0$ and

$$\sup\{|(\varphi_i, x)|: i \geqslant 1\} > (1+\varepsilon)^{-1}|||x||| \quad \text{if } x \in X,$$

since then $\|T\| \geqslant 1$ and $\|T^{-1}\| \leqslant 1 + \varepsilon$ if T is defined by $(Tx)(i) = \varphi_i(x)$ for $i \geqslant 1$. Suppose Δ_n is a positive number and F_n is a finite subset of Φ such that, if $x \in X$, then

$$\sup\{|(f, x)|: f \in F_n \cup \Phi_n\} \geqslant (1 - \Delta_n)|||x|||. \tag{i}$$

If $n = 1$, this is satisfied with $F_1 = \varnothing$ and Δ_1 any positive number. We will show that if $\Delta_{n+1} > \Delta_n$, then there is a finite subset G_n of Φ_n such that, if $x \in X$, then

$$\sup\{|(f, x)|: f \in F_n \cup G_n \cup \Phi_{n+1}\} \geqslant (1 - \Delta_{n+1})|||x|||. \tag{ii}$$

Once this has been done, the existence of $\{\varphi_i\}$ follows by choosing Δ_n uniformly less than $1 - (1 + \varepsilon)^{-1}$, and then letting $F_{n+1} = F_n \cup G_n$ and $\{\varphi_i\} = \bigcup_1^\infty F_n$, arranged in an order so that each member of F_n precedes each member of $G_n = F_{n+1} - F_n$.

Suppose (i) is satisfied, but (ii) is not satisfied, whatever finite subset G_n of Φ_n is used. This implies that, if $\tau = \frac{1}{2}(\Delta_{n+1} - \Delta_n)$ and σ is as described in (B), then for each $k \geqslant 1$ there is an $x_k \in X$ such that $\|x_k\| = 1$ and

$$\sup\{|(f, x)|: f \in F_n \cup G^\tau \cup \Phi^\sigma_{n+1}\} < (1 - \Delta_n - \tau)|||x||| \tag{iii}$$

is satisfied if $x = x_k$, where G^τ is the set of all $\varphi \in \Phi_n$ with $\|\varphi - P_{n+k}\varphi\| \leqslant \frac{1}{3}\tau$, and Φ^σ_{n+1} is the set of all $\varphi \in \Phi_n$ with $|\varphi(n)| \leqslant \sigma$. To see this, observe first that, if G is a finite subset of G^τ for which $P_{n+k}G$ is $\frac{1}{3}\tau$-dense in $P_{n+k}G^\tau$, then G is τ-dense in G^τ. Then the left member of (iii) is not decreased by more than $\tau|||x|||$ if G^τ is replaced by G. Now observe that, for H as given by (B), Φ^σ_{n+1} can be replaced by $H \cup \Phi_{n+1}$ without decreasing the left member of (iii) by more than $\tau|||x|||$ for any x. It now follows that x_k exists, since otherwise (ii) would be satisfied with $G_n = G \cup H$. This x_k has the property

$$\begin{aligned} &\text{if } \varphi \in \Phi_n \text{ and } |(\varphi, x_k)| \geqslant 1 - \Delta_n - \tau, \text{ then } \varphi \notin F_n, |\varphi(n)| > \sigma, \\ &\text{and } \|\varphi - P_{n+k}\varphi\| > \tfrac{1}{3}\tau. \end{aligned} \tag{1}$$

Because of (i) and the fact that x_k satisfies (iii), there is $g_k \in \Phi_n$ for which $|(g_k, x_k)| \geqslant 1 - \Delta_n$. It then follows from (1) that

$$|g_k(n)| > \sigma \quad \text{and} \quad \|g_k - P_{n+k}g_k\| > \tfrac{1}{3}\tau. \tag{2}$$

Now choose a subsequence of $\{g_k\}$ which converges component-wise. Let θ be this component-wise limit, denote the subsequence by $\{\theta + h_k\}$, and let the corresponding members of $\{x_k\}$ be $\{\xi_k\}$. Then (1) is satisfied with x_k replaced by ξ_k. Also,

$$|(\theta + h_k, \xi_k)| \geqslant 1 - \Delta_n. \tag{3}$$

It follows from (2) and the choice of $\{\theta + h_k\}$ that, for each positive integer λ,

$$|\theta(n)| \geqslant \sigma, \quad \lim_{k\to\infty} \|h_k\| \geqslant \tfrac{1}{3}\tau, \quad \text{and} \quad \lim_{k\to\infty} \|P_\lambda h_k\| = 0. \tag{4}$$

It follows from (A) that $\theta/\|\theta\| \in \Phi_n$ and $\operatorname{dist}(h_k/\|h_k\|, \Phi_n) \to 0$. Choose $\lambda > n$ so that $\|\theta - P_\lambda\theta\| < \frac{1}{8}\tau\|\theta\|$. Then it follows from (1) that

$$|(\theta, \xi_k)/\|\theta\|| < 1 - \Delta_n - \tau \quad \text{if } n + k > \lambda. \tag{5}$$

Now choose k so that $n + k > \lambda$, $|h_k(n)| < \frac{1}{2}\sigma\|h_k\|$, $\|P_\lambda h_k\| < \frac{1}{8}\tau$, and $\operatorname{dist}(h_k/\|h_k\|, \Phi_n)$ is small enough that it follows from $|h_k(n)| < \frac{1}{2}\sigma\|h_k\|$ and (1) that

$$|(h_k, \xi_k)/\|h_k\|| < 1 - \Delta_n - \tau + \tfrac{1}{2}\tau/\|h_k\|. \tag{6}$$

Recall that $\|\theta + h_k\| = 1$ and observe that the conditions on λ and k imply

$$\|\theta\| + \|h_k\| \leqslant \|\theta + h_k\| + 2\|\theta - P_\lambda\theta\| + 2\|P_\lambda h_k\| < 1 + \tfrac{1}{2}\tau.$$

Now use this and (5) and (6) to obtain a contradiction of (3):

$$\begin{aligned}|(\theta + h_k, \xi_k)| &< \|\theta\| + \|h_k\| - (\|\theta\| + \|h_k\|)(\Delta_n + \tau) + \tfrac{1}{2}\tau \\ &< 1 + \tfrac{1}{2}\tau - (\Delta_n + \tau) + \tfrac{1}{2}\tau = 1 - \Delta_n.\end{aligned}$$

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