FIXED POINTS IN NONCONVEX DOMAINS

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Abstract. A lemma of Janos is used to prove that nonexpansive self-maps which "shrink" a compact set \( X \) away from its boundary in \( \co X \) have fixed points in \( X \). The lemma is further employed to derive a version, for nonexpansive maps on star-shaped sets, of Janos and Solomon's fixed-point theorem for continuous maps on spaces having an attractor for compact sets.

Janos [2], using a result of A. D. Wallace [4, Theorem 1] in the theory of topological semigroups arrived at the following lemma.

Lemma 1 (Janos [2]). Let \((X, d)\) be a compact metric space and \( f: X \to X \) with \( \{f^n|n \geq 1\} \) equicontinuous. Then there exists a retraction \( r: X \to C_f \) where \( C_f \) denotes the core of \( f \); i.e. \( C_f = \cap \{f^n(X)|n \geq 1\} \).

A particularly significant feature in the proof of this lemma is the demonstration that the retraction \( r \) of \( X \) onto \( C_f \) is in the closure, in the pointwise topology, of the set \( \Gamma_n = \{f^m|m \geq n\} \) for each \( n \geq 1 \) (cf. [2] and [3]). This enables us to state a version of the lemma for nonexpansive maps.

Lemma 2. Let \( T \) be a nonexpansive self-map of a compact metric space \((X, d)\). Then there exists a nonexpansive retraction \( r: X \to C_T \) where

\[
C_T = \cap \{T^n(X)|n \geq 1\}.
\]

Proof. The family \( \{T^n\} \) is equicontinuous, for given any \( \epsilon > 0 \) if we merely choose \( \delta = \epsilon \) then \( d(x, y) < \delta \) implies that \( d(T^n x, T^n y) < d(x, y) \) for all \( n \geq 1 \). Now the retraction \( r \) guaranteed by Lemma 1 is the pointwise limit of mappings of the form \( T^n \). Thus, for any \( x, y \in X \) and any \( \epsilon > 0 \) there exists an \( n \) such that \( d(rx, T^n x) \leq \epsilon \) and \( d(T^n y, ry) \leq \epsilon \). Hence

\[
d(rx, ry) \leq d(rx, T^n x) + d(T^n x, T^n y) + d(T^n y, ry) < d(T^n x, T^n y) + 2\epsilon \leq d(x, y) + 2\epsilon.
\]

Since \( \epsilon \) was arbitrary this shows \( d(rx, ry) < d(x, y) \) and thus \( r \) is a nonexpansive mapping.

A normed linear space \( X \) is said to be strictly convex if whenever \( x, y \in X \) with \( x \neq y \), \( \|x\| = \|y\| = S \) and \( 0 < \alpha < 1 \), then \( \|\alpha x + (1 - \alpha)y\| < S \). A useful property of strictly convex spaces is that for any three noncollinear points \( x, y, z \in X \), the triangle inequality is strict; i.e. \( d(x, z) < d(x, y) + d(y, z) \).

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In the results which follow, we shall denote by $\partial' X$ the boundary of $X$ in the closed convex hull of $X$.

**Theorem 1.** Let $X$ be a compact subset of a strictly convex normed linear space $E$ and let $T: X \to X$ be nonexpansive. If there exists an $n > 1$ such that $T^n X \cap \partial' X = \emptyset$ then $T$ has a fixed point in $X$.

**Proof.** As before let $C_T = \bigcap_{k=1}^{\infty} T^k X$. By Lemma 2 there exists a nonexpansive retraction $r: X \to C_T$. Suppose $C_T$ is not convex. Then there exists $x, y \in C_T$ such that $x'y = \{ax + (1 - a)y | 0 < a < 1\}$ is not contained entirely in $C_T$. Let $a = \sup\{\beta | ax + (1 - a)y \in C_T, 0 < a < \beta\}$. Since $C_T$ is compact, $z = ax + (1 - a)y$ is in $C_T$. Now if $x'y$ contains no points of $X \setminus C_T$ then $z$ is a $\partial'$-boundary point of $X$ in $C_T \subseteq T^n X$ contradicting the hypothesis. Hence there exists a point $w \in x'y \cap \{X \setminus C_T\}$. But then we shall have

$$\|rx - rw\| + \|rw - ry\| < \|x - w\| + \|w - y\| = \|x - y\|$$

and, since $rx = x, ry = y$,

$$\|x - rw\| + \|rw - y\| < \|x - y\|.$$ 

But $rw \in C_T$ and so $w \neq rw$. The mapping $r$ is nonexpansive and so $rw$ is not collinear with $x, y$ for if $w = bx + (1 - b)y$ and $rw = cx + (1 - c)y$ we shall have $b \neq c$, say, $c < b < 1$. But then

$$\|x - w\| = (1 - b)\|x - y\|$$

and

$$\|rx - rw\| = \|x - rw\| = (1 - c)\|x - y\|$$

and since $(1 - c) > (1 - b)$ we would have $\|x - w\| < \|rx - rw\|$, a contradiction. If, on the other hand, $0 < b < c$ then $\|y - w\| = b\|x - y\|$ and $\|ry - rw\| = \|y - w\| = c\|x - y\|$ and so $\|y - w\| < \|ry - rw\|$, another contradiction. Thus, $x, y, rw$ are not collinear. But then $\|x - rw\| + \|rw - y\| = \|x - y\|$ contradicting an earlier inequality. Thus $C_T$ must be convex. Now $T: C_T \to C_T$ and so $T$ has a fixed point in $C_T$ by Schauder's Theorem.

**Corollary 1.** Let $X$ be a compact subset of a strictly convex normed linear space $E$ and suppose $T, g$ are self-maps of $X$ such that $g$ is continuous, $T$ is nonexpansive and $T^n X \cap \partial' X = \emptyset$ for some $n > 1$. If $Tg = gT$ then $T, g$ have a common fixed point in $X$.

**Corollary 2.** Let $X$ be a compact subset of a strictly convex normed linear space $E$. Suppose that $\{T_a\}$ is a family of commuting, nonexpansive self-maps of $X$ and that for one $T \in \{T_a\}$, there exists an $n > 1$ such that $T^n X \cap \partial' X = \emptyset$. Then the family $\{T_a\}$ has a common fixed point in $X$.

The proofs of these corollaries utilize standard techniques.

The suspicion that "compact" may not be replaced by "closed and bounded" in the hypotheses of Theorem 1 is easily confirmed by the following example in the Hilbert space $l_2$: Let $e_n$ be the $n$th unit vector in $l_2$ and let
\[ X = \bigcup_{n=1}^{\infty} B(e_n, n/(2n+1)). \]

Then \( X \) is closed and bounded. Define the mapping \( T: X \to X \) by
\[ T(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots). \]
Then \( T \) is nonexpansive and \( TX \cap \partial X = \emptyset \). But \( T \) has no fixed point in \( X \).

**Definition.** Let \( X \) be a topological space and \( f: X \to X \) a map. A nonempty subset \( M \subseteq X \) is called an "attractor for compact sets under \( f \)" if (a) \( M \) is compact and \( fM \subseteq M \), (b) for any compact set \( G \subseteq X \) and open set \( 0 \) containing \( M \), there exists an \( N = N(G, 0) \) such that \( f^NG \subseteq 0 \) for all \( n \geq N \).

Janos and Solomon [3], using Lemma 1, have proven

**Theorem 2.** Let \( f: G \to G \), \( G \) a closed, convex subset of a Banach space, \( f \) a mapping satisfying:

(i) there exists \( M \subseteq G \) an attractor for compact sets under \( f \), and
(ii) the family \( \{ f^n \}_{n=1}^{\infty} \) is equicontinuous.

Then \( f \) has a fixed point in \( X \).

We shall prove a result analogous to this theorem for nonexpansive maps on star-shaped sets. The proof parallels that of Janos and Solomon.

**Theorem 3.** Let \( S \) be a closed, star-shaped subset of a Banach space \( E \) and \( T: S \to S \) nonexpansive. If there exists \( M \subseteq S \), an attractor for compact sets, then \( T \) has a fixed point in \( S \).

**Proof.** Let \( p \) be a star-point of \( S \) and define the closed star-hull of \( M \) to be
\[ \text{sh} M = \text{cl}\{ tp + (1-t)x | 0 < t < 1, x \in M \}. \]
\( M \) is compact, so \( M \cup \{ p \} \) is compact. Now the convex hull of a compact set is compact, so \( \text{co}[M \cup \{ p \}] \) is compact. But \( \text{sh} M \subseteq \text{co}[M \cup \{ p \}] \) so \( \text{sh} M \) is compact. Let
\[ X = \text{cl}\{ \bigcup_{n=0}^{\infty} T^n(\text{sh} M) | n > 0 \}. \]

Since \( T[T^n(\text{sh} M)] = T^{n+1}(\text{sh} M) \) for all \( n > 0 \) and \( T \) is continuous, \( T: X \to X \). \( X \) is compact, for if \( 0_{\epsilon/2} \) is the \( \epsilon/2 \)-parallel set of \( M \) (i.e. \( 0_{\epsilon/2} = \bigcup_{x \in M} S(x, \epsilon/2) \)) there exists an \( N \) such that \( n > N \) implies \( T^n(\text{sh} M) \subseteq 0_{\epsilon/2} \). But \( M \) compact implies that \( M \) has a finite \( \epsilon/2 \)-mesh. Thus \( 0_{\epsilon/2} \) has a finite \( \epsilon \)-mesh. So \( \bigcup_{n>0} T^n(\text{sh} M) \) has a finite \( \epsilon \)-mesh and since \( \bigcup_{n>N} T^n(\text{sh} M) \) is compact, \( X \) is itself compact.

Now let \( C_T = \bigcap_{n=0}^{\infty} T^n(X) \). \( C_T \) is nonempty and compact.

\[ TC_T = T \bigcap_{n=0}^{\infty} T^n(X) = \bigcap_{n=0}^{\infty} T^{n+1}(X) = C_T, \]

so by the definition of \( M \), \( C_T \subseteq M \). By Lemma 2 there exists a nonexpansive retraction \( r: X \to C_T \). The map \( Tr: \text{sh} M \to C_T \) is a nonexpansive self-map of a compact star-shaped set \( \text{sh} M \) and so by Dotson's theorem [1] has a fixed point \( x \in \text{sh} M \). Now \( r(x) \in C_T \) and \( T: C_T \to C_T \) so \( x = Tr(x) \in C_T \). But then \( r(x) = x \) and thus \( Tx = x \).
References


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