PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 4, December 1980

BEST APPROXIMATION IN CERTAIN DOUGLAS ALGEBRAS

RAHMAN YOUNIS

ABSTRACT. The main result of this paper is that if $f \in L^\infty$ and S is a weak peak set for H^∞, then the distance from f to the Douglas algebra $\{f \in L^\infty: f|_S \in H^\infty|_S\}$ is attained.

1. Introduction. Let L^∞ denote the usual Lebesgue space of functions on the unit circle. Let H^∞ denote the subalgebra of boundary values of bounded analytic functions on $|z| < 1$. Let $H^\infty + C$ denote the closed linear span of H^∞ and C, where C is the set of continuous complex-valued functions on $|z| = 1$.

The authors of [3] prove the following theorem.

THEOREM A. *If $f \in L^\infty$, then* $\operatorname{dist}(f, H^\infty + C) = \|f - g\|_\infty$ *for some g in $H^\infty + C$.*

In [14], D. Luecking proves the following theorem.

THEOREM B. *$(H^\infty + C)/H^\infty$ is an M-ideal in L^∞/H^∞.*

Theorem B implies Theorem A. The author of [14] asked whether in Theorem B, $H^\infty + C$ can be replaced by other subalgebras. Also, D. Sarason [15, p. 113] and the authors of [3, p. 609] asked whether in Theorem A, $H^\infty + C$ can be replaced by an arbitrary Douglas algebra.

In this paper, we prove a theorem (Theorem 3.1) which answers the question raised in [14], and its Corollary 3.2 gives a partial solution to the question raised in [3] and [15]. Theorem 4.1 is related to the general L^∞ distance problem.

2. Preliminaries. We identify L^∞ with $C(X)$, via the Gelfand transform, where X is the maximal ideal space of L^∞. Thus H^∞ can be considered as a function algebra on X. No notational distinction will be made between f in L^∞ viewed as a function on the unit circle and its Gelfand transform $\hat{f}$ viewed as a continuous function on X.

A subset S of X is called a peak set for H^∞ if there exists a function f in H^∞ such that $f = 1$ on S and $|f| < 1$ off S. We say that E is a weak peak set for H^∞ if E is the intersection of some collection of peak sets of H^∞. If S is a weak peak set for H^∞, then $H^\infty|_S$ is closed in $L^\infty|_S$ [4, p. 104]. A subset E of X is called an antisymmetric set for a closed subalgebra B ($H^\infty \subset B \subset L^\infty$) if $f \in B$; f is real on E implies that f is constant on E. A maximal antisymmetric set for B is a closed

Received by the editors November 28, 1979.
1980 *Mathematics Subject Classification.* Primary 30H05, 46E15; Secondary 41A50.
Key words and phrases. M-ideal, weak peak set, best approximation.

© 1980 American Mathematical Society
0002-9939/80/0000-0621/$02.00

antisymmetric set which is not contained properly in any closed antisymmetric set for B. The reader is referred to [12] for many of the properties of H^∞ and L^∞ and to [4] and [9] for additional properties of function algebras.

DEFINITION. A subspace J of a Banach space Y is called an L-ideal if there is a projection E of Y onto J such that $\|y\| = \|Ey\| + \|y - Ey\|, y \in Y$. Such an E is called an L-projection.

A subspace K of a Banach space X is called an M-ideal if the annihilator $K^\perp$ is an L-ideal of X^* (see [1] for these concepts).

The following properties of an M-ideal of X are needed in this paper.

LEMMA C [1, COROLLARY 5.6]. *If K is an M-ideal of X and if $x \in X$, then there exists $m \in K$ such that* $\operatorname{dist}(x, K) = \|x - m\|$.

LEMMA D [13, THEOREM 3]. *If K is an M-ideal of X and if $x \in X$, then the span of $P_K(x) = K$ if $x \notin K$, where $P_K(x) = \{m \in K\colon \operatorname{dist}(x, K) = \|x - m\|\}$.*

3. The main results. Let S be a subset of X, the maximal ideal space of L^∞, which is a weak peak set for H^∞. Define $H_S^\infty = \{f \in L^\infty\colon f|_S \in H^\infty|_S\}$. It is a closed subalgebra of L^∞ which contains H^∞.

THEOREM 3.1. *H_S^∞/H^∞ is an M-ideal in L^∞/H^∞.*

COROLLARY 3.2. *If $f \in L^\infty$, then* $\operatorname{dist}(f, H_S^\infty) = \|f - h\|_\infty$ *for some h in H_S^∞.*

PROOF OF COROLLARY 3.2. Theorem 3.1 together with Lemma C imply that for any f in L^∞, there exists g in H_S^∞ such that $\operatorname{dist}(f, H_S^\infty) = \operatorname{dist}(f - g, H^\infty)$. Since H^∞ is weak star closed in L^∞, a compactness argument yields g_0 in H^∞ such that $\operatorname{dist}(f - g, H^\infty) = \|f - g - g_0\|_\infty$. Set $h = g + g_0$; then $\operatorname{dist}(f, H_S^\infty) = \|f - h\|_\infty$.

COROLLARY 3.3. *If $S \neq X$ and $f \notin H_S^\infty$ then the best approximation h in Corollary* 3.2 *is not unique.*

Indeed, if $f \notin H_S^\infty$, then by Lemma D, h is never unique.

PROOF OF THEOREM 3.1. We identify L^∞ with $C(X)$, $(L^\infty/H^\infty)^*$ with

$$(H^\infty)^\perp = \left\{ \mu \in C(X)^*\colon \int f\,d\mu = 0, f \in H^\infty \right\} \quad \text{and}$$

$$(H_S^\infty/H^\infty)^\perp = \left\{ \mu \in C(X)^*\colon \int f\,d\mu = 0, f \in H_S^\infty \right\}.$$

To prove the theorem, we have to produce an L-projection E of $(H^\infty)^\perp$ onto $(H_S^\infty)^\perp$.

Let $\mu \perp H^\infty$. Define $E\mu = \chi_S\mu$, where χ_S denotes the characteristic function of S. By [4, p. 106], $\chi_S\mu \perp H^\infty$. It is easy to see that $H_S^\infty = H^\infty + J_S$, where $J_S = \{f \in L^\infty\colon f(S) = 0\}$. In order to show that $\chi_S\mu \perp H_S^\infty$, it suffices to show that $\chi_S\mu \perp J_S$. But for f in J_S, $\int f\chi_S\,d\mu = 0$. Thus $E\mu \perp H_S^\infty$. Note that $E^2\mu = E(\chi_S\mu) = \chi_S^2\mu = \chi_S\mu = E\mu$. Moreover, $\|\mu\| = \|\chi_S\mu + (1 - \chi_S)\mu\| = \|E\mu\| + \|\mu - E\mu\|$.

Finally, we have to show that E is onto. Let $\mu \perp H_S^\infty$. Then $\mu \perp H^\infty + \mu \perp J_S$. We claim that the support of μ ($=$ supp μ) is contained in S. To see that, suppose there exists $x \in X \setminus S$ such that $x \in \operatorname{supp} \mu$. Let W be a clopen set of X such that $x \in W$ and $W \cap S = \varnothing$. Now

$$\int \chi_W \, d|\mu| = \sup\left\{\left|\int_X g \, d\mu\right| : g \in L^\infty, 0 < |g| < \chi_W\right\}.$$

(See [11, p. 364].) Since $\chi_W = 0$ on S, we have $g = 0$ on S. Consequently, $|\mu|(W) = \int \chi_W \, d|\mu| = 0$. This contradiction shows that $\operatorname{supp} \mu$ lies in S. Hence $(1 - \chi_S)\mu \equiv 0$. That is $\mu = \chi_S \mu$. Thus $E_\mu = \chi_S \mu = \mu$. This ends the proof of the theorem.

4. The general distance problem. Let B be an arbitrary closed subalgebra of L^∞ which contains H^∞. If S is a maximal antisymmetric set for B then $H^\infty|_S = B|_S$ [2, p. 20]. Since S is a weak peak set for B, by [4, p. 104] $B|_S$ is closed in $L^\infty|_S$. Since $H^\infty|_S$ is closed in $L^\infty|_S$, this forces S to be a weak peak set for H^∞ [9, p. 65].

THEOREM 4.1. *Let B be a closed subalgebra of L^∞ and f be in L^∞. Then for every maximal antisymmetric set S for B, there exists h in H_S^∞ such that* $\operatorname{dist}(f, B) = \|f - h\|_\infty$.

We need the following lemma.

LEMMA E [7]. *Let $u \in L^\infty$ and S be a weak peak set for H^∞. Then* $\operatorname{dist}(u, H_S^\infty) = \operatorname{dist}_S(u, H^\infty)$, *where* $\operatorname{dist}_S(u, H^\infty) = \inf\{\|u - h\|_S : h \in H^\infty\}$.

PROOF OF THEOREM 4.1. Using [10, p. 419], there exists a maximal antisymmetric set E of B such that $\operatorname{dist}(f, B) = \operatorname{dist}(f|_E, B|_E)$. Thus

$$\operatorname{dist}(f, B) = \operatorname{dist}(f|_E, B|_E) = \operatorname{dist}(f|_E, H^\infty|_E) = \operatorname{dist}_E(f, H^\infty).$$

Hence by Lemma E, we have $\operatorname{dist}(f, B) = \operatorname{dist}(f, H_E^\infty)$.

Let S be an arbitrary maximal antisymmetric set for B. Then $B \subset H_S^\infty \cap H_E^\infty \subset H_E^\infty$. Since $\operatorname{dist}(f, B) = \operatorname{dist}(f, H_E^\infty)$, we see that $\operatorname{dist}(f, B) = \operatorname{dist}(f, H_S^\infty \cap H_E^\infty)$. We claim that $H_S^\infty \cap H_E^\infty = H_{S \cup E}^\infty$. Assume the claim for a moment; then $\operatorname{dist}(f, H_{S \cup E}^\infty) = \operatorname{dist}(f, B)$. By Theorem 3.1, there exists $h \in H_{S \cup E}^\infty$ such that $\operatorname{dist}(f, B) = \|f - h\|_\infty$.

Now to prove the claim we note that $H_S^\infty \cap H_E^\infty$ contains $H_{S \cup E}^\infty$. To prove the reverse, it suffices, according to the Chang-Marshall theorem [6], to prove that any inner function which is invertible in $H_S^\infty \cap H_E^\infty$ is also invertible in $H_{S \cup E}^\infty$. Let b be an inner function such that $\bar{b}|_S \in H^\infty|_S$ and $\bar{b}|_E \in H^\infty|_E$. Then $b(S) = a$ and $b(E) = c$, where a and c are constant numbers such that $|a| = 1$ and $|c| = 1$. Let Q be a finite Blaschke product such that $Q(a) = \bar{a}$ and $Q(c) = \bar{c}$ (see [5], [16]). Then $(Q \circ b)|_{S \cup E} = \bar{b}|_{S \cup E}$. This shows that $\bar{b} \in H_{S \cup E}^\infty$. This proves the claim, and consequently this ends the proof of Theorem 4.1.

REMARKS. (1) The proof of Theorem 4.1 shows that if $S_1, \ldots, S_N$ are maximal antisymmetric sets for B, then there exists h in $\bigcap_{i=1}^N H_{S_i}^\infty$ such that $\operatorname{dist}(f, B) = \|f - h\|_\infty$. The author does not know if h can be chosen to be in $\bigcap_S H_S^\infty = B$, where S runs over all maximal antisymmetric sets for B.

(2) Perhaps, one might think that Theorem 3.1 implies Theorem A. The answer is no. In fact, there does not exist a weak peak set S for H^∞ such that $H^\infty + C = H_S^\infty$. To see that, assume that $H^\infty + C = H_S^\infty$ for some S. Let W be a clopen set in X which contains S. Then $\chi_W \in H_S^\infty$. Thus $\chi_W \in H^\infty + C$. This is impossible because the maximal ideal space of $H^\infty + C$ is connected (see [**8**, Corollary 6.42] and [**12**, p. 188]).

Acknowledgements. I would like to thank D. Patil and P. Ahern for helpful discussions.

References

1. E. Alfsen and E. Effros, *Structure in real Banach spaces*, Ann. of Math. (2) **96** (1972), 98–173.

2. S. Axler, *Subalgebras of L^∞*, Thesis, University of California at Berkeley, 1975.

3. S. Axler, I. D. Berg, N. Jewell and A. Shields, *Approximation by compact operators and the space $H^\infty + C$*, Ann. of Math. (2) **109** (1979), 601–612.

4. A. Browder, *Introduction to function algebras*, Benjamin, New York, 1969.

5. D. G. Cantor and R. R. Phelps, *An elementary interpolation theorem*, Proc. Amer. Math. Soc. **16** (1965), 523–525.

6. S. Y. Chang and D. E. Marshall, *Some algebras of bounded analytic functions*, Lecture Notes in Math., vol. 604, Springer-Verlag, New York, 1977, pp. 12–20.

7. K. Clancey and J. Gosselin, *On the local theory of local Toeplitz operators*, Illinois J. Math. **22** (1978), 449–458.

8. R. G. Douglas, *Banach algebra techniques in operator theory*, Academic Press, New York, 1972.

9. T. Gamelin, *Uniform algebras*, Prentice-Hall, Englewood Cliffs, N.J., 1969.

10. I. Glicksberg, *Measures orthogonal to algebras and sets of antisymmetry*, Trans. Amer. Math. Soc. **105** (1962), 415–435.

11. E. Hewitt and K. Stromberg, *Real and abstract analysis*, Springer-Verlag, New York, 1975.

12. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall Studies in Modern Analysis, Prentice-Hall, Englewood Cliffs, N.J., 1962.

13. R. Holmes, B. Scrantan and J. Ward, *Approximation from the space of compact operators and other M-ideals*, Duke Math. J. **42** (1975), 259–269.

14. D. H. Luecking, *The compact Hankel operators form an M-ideal in the space of Hankel operators*, Proc. Amer. Math. Soc. **79** (1980), 222–224.

15. D. Sarason, *Lecture notes on function theory of the unit circle*, Virginia Polytechnic and State University, Blacksburg, Virginia, June 19-23, 1978.

16. R. Younis, *Interpolation by a finite Blaschke product*, Proc. Amer. Math. Soc. **78** (1980), 451–452.

Department of Mathematics, University of Wisconsin, Milwaukee, Wisconsin 53201