PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 4, December 1980

ON OPERATOR RANGES

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ABSTRACT. If f is any vector-valued bounded function defined on open set D of the complex plane, and T is any bounded linear operator on a Hilbert space such that $\sigma_R(T^*)$ is empty and if $(T - zI)^2 f(z) = x$ for all z in D then f is analytic.

Let T be a bounded linear operator on a Hilbert space H. Let f be an H-valued function defined on an open set D of the complex plane. Suppose $(T - zI)f(z) = x$ for all z in D and for a fixed x in H. The question arises: What type of conditions on the operator T and on the function f will be sufficient to insure that f is analytic on D? This question has been implicitly discussed in Clancey [1], Johnson [4], Putnam [5], [6], Radjabalipour [7], and Stampfli-Wadhwa [9], [10], under various conditions related to normality on T. One of the main results is: If T is a hyponormal operator ($TT^* < T^*T$) and f is any function satisfying the above equation then f is analytic in D. See [1]. On the other hand, there is a cohyponormal operator T, (T^* is hyponormal) and a bounded function f satisfying the above equation which fails to be analytic on D. See [6].

In this note, by a simple argument, we shall show that if $(T - zI)^2 f(z) = x$ for all z in D and if f is bounded on D and $\sigma_R(T^*) = \varnothing$ (range of $T^* - \bar{z}I$ is dense in H for all z) then f is analytic. We shall use this result to give an alternative proof of a classical result of Stampfli [8] about quadratically hyponormal operators.

The following lemma is implicitly contained in most of the references mentioned previously. We include it for the sake of completeness.

LEMMA. *Let $(T - zI)f(z) = x$ for all z in D be such that f is bounded and $\sigma_R(T^*) = \varnothing$ then f is weakly continuous.*

PROOF. For any z and z_0 in D,

$$\begin{aligned}(f(z) - f(z_0), (T^* - \bar{z}_0 I)y) &= ((T - z_0 I)(f(z) - f(z_0)), y) \\ &= (z - z_0)(f(z), y) \quad \text{for all } y \text{ in } H.\end{aligned}$$

Since f is bounded and range of $(T^* - \bar{z}_0 I)$ is dense in H, it follows that f is weakly continuous.

THEOREM. *Let g be a bounded vector-valued function such that $(T - zI)^2 g(z) = x$ for all z in D and let $\sigma_R(T^*) = \varnothing$. Then g is analytic on D.*

PROOF. Let $f(z) = (T - zI)g(z)$. Then $(T - zI)f(x) = x$ for all z in D, f is bounded and hence weakly continuous on each bounded subset D_0 of D. Now for

Received by the editors October 30, 1979 and, in revised form, January 29, 1980.
AMS (MOS) subject classifications (1970). Primary 47A05, 47B20.

any z and z_0 in D_0,

$$((f(z) - f(z_0))/(z - z_0), (T^* - \bar{z}_0 I)y) = (f(z), y),$$

and

$$\lim_{z \to z_0} ((f(z) - f(z_0))/(z - z_0), (T^* - \bar{z}_0 I)y) = (f(z_0), y)$$
$$= ((T - z_0 I)g(z_0), y) = (g(z_0), (T^* - \bar{z}_0 I)y)$$

for all y in H.

Since range of $(T^* - \bar{z}_0 I)$ is dense in H, f is analytic and $f'(z) = g(z)$ for all z in D.

Let T be a quadratically hyponormal operator ($aT^2 + bT + cI$ is hyponormal for all complex numbers a, b and c). Let $\rho(T, x)$ be the local resolvent of the vector x with respect to the operator T (see Dunford and Schwartz [3, p. 1935]).

COROLLARY (STAMPFLI [8]). *If T is a quadratically hyponormal operator with $\sigma_R(T) = \varnothing$ then $\rho(T; x) \subset \overline{\rho(T^*, x)}$. (The bar denotes the complex conjugate of the set.)*

PROOF. Let $z_0 \in \rho(T, x)$; thus there exists an analytic function defined on a bounded set D containing z_0 such that $(T - zI)f(z) = x$. Since f is analytic, a simple computation shows that $(T - zI)^2 f'(z) = x$ for all z in D. Since T is quadratically hyponormal, $(T - zI)^2(T^* - \bar{z}I)^2 < (T^* - \bar{z}I)^2(T - zI)^2$. By Douglas [2], there is a contraction $K(z)$ such that $(T - zI)^2 = (T^* - \bar{z}I)^2 K(z)$. Consequently, $(T^* - \bar{z}I)^2 g(\bar{z}) = x$ where $g(\bar{z}) = K(z)f'(z)$. Thus $g(\bar{z})$ is bounded for $z \in D$. Using the Theorem we conclude that $g(z)$ is analytic for all z in $\bar{D}$ and $(T^* - zI)(T^* - zI)g(z) = x$ for all z in $\bar{D}$; thus $z_0 \in \overline{\rho(T^*, x)}$.

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