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A NOTE ON ANALYTIC SETS

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ABSTRACT. We give "effective" proofs of two recent theorems on analytic sets of reals, together with counterexamples to their natural extensions.

In this note we give effective ("lightface") proofs of two recent theorems about analytic (Σ_1^1) sets of reals due to C. Dellacherie, together with counterexamples to their natural extensions. All our proofs are quite easy modulo the basics of the effective theory.

In the case $k = 2$, the following is just the Reduction Theorem for Π_1^1 sets of reals.

THEOREM 1 (DELLACHERIE [1]). *Let* $\{A_i | i < k\}$ *be a finite family of* Π_1^1 *subsets of* ${}^\omega\omega$. *Then there are* Π_1^1 *sets* B_i, $i < k$, *so that*

(i) $B_i \subseteq A_i$ *for* $i < k$;
(ii) $B_i \cup B_j = A_i \cup A_j$ *for* $i \neq j$;
(iii) $\bigcap_{i<k} B_i = \varnothing$.

PROOF. Let $R(i, x)$ iff $x \in A_i$. Let φ: $R \to \omega_1$ be a Π_1^1 norm on R (cf. [3]) and put $\varphi(i, x) = \omega_1$ if $\neg R(i, x)$. Let

$$x \in B_i \Leftrightarrow R(i, x) \wedge \exists j < k(\varphi(i, x) < \varphi(j, x) \vee (\varphi(i, x) = \varphi(j, x) \wedge i < j)).$$

Then B_i is Π_1^1. Note that if $\exists i < k(x \notin A_i)$ then $\forall i < k(x \in A_i \Leftrightarrow x \in B_i)$, while if $\forall i < k(x \in A_i)$ then $x \notin B_i$ for exactly one $i < k$. Thus the B_i are as desired. $\square$

Dellacherie asked whether Theorem 1 holds for countable families. A. S. Kechris and the author found a counterexample, which L. Harrington simplified considerably to the following. (Apparently, Kunen had earlier found a counterexample using forcing.)

COUNTEREXAMPLE 1. Let

$$x \in A_0 \Leftrightarrow \text{codes an } \omega\text{-sequence } \{C_n^x | n < \omega\} \text{ of } \mathbf{\Pi}_1^1 \text{ subsets of } {}^\omega\omega;$$
$$x \in A_{n+1} \Leftrightarrow x\, A_0 \wedge x \in C_n^x.$$

Suppose $\{B_n | n < \omega\}$ were an ω-sequence of $\mathbf{\Pi}_1^1$ sets reducing $\{A_n | n < \omega\}$ in the sense of Theorem 1. Let x code $\{B_n | n < \omega\}$; i.e. let $B_n = C_n^x$ for $n < \omega$. By (iii) let n be least so that $x \notin B_n$. Then $x \in A_m$ for $m \leqslant n$ by the definition of $\{A_n | n < \omega\}$. So $x \in B_{n+1}$ by (ii). But then $x \in A_{n+1}$, so $x \in C_n^x = B_n$, a contradiction. $\square$

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If $R \subseteq A \times B$ and $x \in A$, we let $R_x = \{y \in B | (x, y) \in R\}$. R_x is called a section of R.

THEOREM 2 (DELLACHERIE [1]). *Let* $R \subseteq {}^\omega\omega \times {}^\omega\omega$ *be* Σ^1_1, *and* R_x *compact for each* $x \in {}^\omega\omega$. *Then there is a* Σ^1_1 *set* $S \subseteq {}^\omega\omega \times {}^\omega\omega$ *so that*

(a) $\forall x(S_x$ *is compact and nonempty*);

(b) $\forall x(R_x \neq \varnothing \Rightarrow R_x = S_x)$.

PROOF. Let R be as above. Define R^* by

$$R^*(x, s) \Leftrightarrow s \in \omega^{<\omega} \wedge \exists y(s \subseteq y \wedge R(x, y)).$$

Then R^* is Σ^1_1, and R^*_x is a tree on ω for each x. Since R_x is closed, R_x is just the set of branches of R^*_x; since R_x is compact, R^*_x is finitely branching. An easy boundedness argument yields a Δ^1_1 set $T^* \supseteq R^*$ so that T^*_x is an infinite, finitely branching tree on ω for each x.

Let φ be a Π^1_1 norm on $({}^\omega\omega \times \omega^{<\omega}) - R^*$ and define

$$\psi(x, s) = \min\{\varphi(x, t) | t \subseteq s\}.$$

Then ψ is a Π^1_1 norm on $({}^\omega\omega \times \omega^{<\omega}) - R^*$, and $\{s | \psi(x, s) > \alpha\}$ is a tree for each x, α. Let

$$S^*(x, s) \Leftrightarrow T^*(x, s) \wedge (R^*(x, s) \vee T^*_x - \{t | \psi(x, t) < \psi(x, s)\} \text{ is finite}).$$

S^*_x is a finitely branching tree for each x, and $S^*_x = R^*_x$ if R^*_x is finite (i.e., if $R_x \neq \varnothing$). If R^*_x is finite, let α be least so that $T^*_x - \{t | \psi(x, t) < \alpha\}$ is finite. Since T^*_x is finitely branching, $\alpha = \beta + 1$ for some β. But then $\psi(x, s) = \beta$ for infinitely many s, and $\psi(x, s) = \beta \Rightarrow S^*(x, s)$ by the definition of S^*. Thus S^*_x is an infinite finitely branching tree for each x. Let $S(x, y) \Rightarrow y$ is a branch of S^*_x. $\square$

Dellacherie asked whether Theorem 2 holds with "σ-compact" replacing "compact".

COUNTEREXAMPLE 2. Let $\{A_n | n < \omega\}$ be as defined in Counterexample 1. Let

$$R(x, y) \Leftrightarrow \exists n(\forall i(y(i) = n) \wedge x \notin A_n).$$

Then R is Σ^1_1 and R_x is countable, hence σ-compact, for each x. The argument of Counterexample 1 shows that if $S \subseteq {}^\omega\omega \times {}^\omega\omega$ is Σ^1_1 and $\forall x(R_x \neq \varnothing \Rightarrow R_x = S_x)$, then $\exists x(R_x = \varnothing \wedge S_x = \varnothing)$. $\square$

The following is an old theorem.

THEOREM 3 (LUSIN [2]). *Let* $R \subseteq {}^\omega\omega \times {}^\omega\omega$ *be* Σ^1_1 *and* R_x *countable for each* x. *Then there are* Σ^1_1 *sets* S_n, $n < \omega$, *so that*

(i) $R = \bigcup_n S_n$;

(ii) *for each* x, n, $(S_n)_x$ *has* $\leqslant 1$ *element*.

Dellacherie asked whether Theorem 3 holds with "σ-compact" replacing "countable" and "is compact" replacing "has $\leqslant 1$ element".

COUNTEREXAMPLE 3. For $x \in {}^\omega\omega$, let $W^x = \{e \in \omega | e$ is the Gödel number of a well-order of ω recursive in $x\}$. The relation "$y = W^x$" is Π^1_1, as is "$y = W^{W^x}$". Define now

$$R(x, y) \Leftrightarrow y \in {}^\omega 2 \wedge y \neq W^{W^x}.$$

Then R is Σ_1^1, and R_x is σ-compact (in fact, open relative to ${}^\omega 2$) for each x. Suppose $\{S_n | n < \omega\}$ were a family of $\boldsymbol{\Sigma}_1^1$ sets as in the proposed extension of Theorem 3. Say that "$x \in S_n$" is a $\Sigma_1^1(z)$ relation of x, n. Let $S_n^* = \{s \in \omega^{<\omega} | \exists y(s \subseteq y \wedge y \in S_n)\}$. The sequence $\langle S_n^* | n < \omega \rangle$ is recursive in W^z. But W^{W^z} is the unique $y \in {}^\omega 2$ so that y is not a branch of any of the trees S_n^*. Thus W^{W^z} is Δ_1^1 in W^z, a contradiction. $\square$

Counterexample 3 raises the following question: let $R \subseteq {}^\omega\omega \times {}^\omega\omega$ be Σ_1^1 with σ-compact sections. Must there be $S \subseteq R$, Σ_1^1 with compact sections, so that $R_x \neq \varnothing \Rightarrow S_x \neq \varnothing$? Clearly this is one instance of a family of problems generalizing the uniformization problem. If R, $S \subseteq {}^\omega\omega \times {}^\omega\omega$ we say S almost uniformizes R iff $S \subseteq R$ and $\forall x(R_x \neq \varnothing \Rightarrow S_x \neq \varnothing)$. Notice that if R is Σ_1^1, then there is a Σ_1^1 S with Borel sections which almost uniformizes R; let $Sxy \Leftrightarrow (Rxy \wedge \omega_1^x = \omega_1^{x,y})$, where ω_1^z is the least ordinal not recursive in z. On the other hand, L. Harrington has found a Σ_1^1 relation R such that, for every countable α, R cannot be almost uniformized by a $\boldsymbol{\Sigma}_1^1$ relation with $\boldsymbol{\Pi}_\alpha^0$ sections. We sketch his example for those familiar with [4]. Let $\mathbf{P}$ be the forcing notion of §2 of [4]. Let $R(x, T)$ iff T is $\mathbf{P}$-generic over $L_{\omega_1^x}[x]$ with respect to Σ_1 sentences. Then R is Σ_1^1, since $R(x, T)$ iff $\omega_1^x = \omega_1^{x,T}$ and T is $\mathbf{P}$-generic over $L_{\omega_1^x}[x]$ with respect to ranked sentences. Lemma 1 of [4] implies that R cannot be almost uniformized by a $\boldsymbol{\Sigma}_1^1$ relation with sections of bounded Borel class.

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