A NOTE ON ANALYTIC SETS

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ABSTRACT. We give "effective" proofs of two recent theorems on analytic sets of reals, together with counterexamples to their natural extensions.

In this note we give effective ("lightface") proofs of two recent theorems about analytic (Σ₁¹) sets of reals due to C. Dellacherie, together with counterexamples to their natural extensions. All our proofs are quite easy modulo the basics of the effective theory.

In the case k = 2, the following is just the Reduction Theorem for Π¹¹ sets of reals.

THEOREM 1 (DELLACHERIE [1]). Let \( \{A_i|i \leq k\} \) be a finite family of Π¹¹ subsets of \( \omega \). Then there are Π¹¹ sets \( B_i, i \leq k \), so that

(i) \( B_i \subseteq A_i \) for \( i < k \);
(ii) \( B_i \cup B_j = A_i \cup A_j \) for \( i \neq j \);
(iii) \( \bigcap_{i \leq k} B_i \) = \( \emptyset \).

Proof. Let \( R(i, x) \) iff \( x \in A_i \). Let \( \varphi: R \to \omega \) be a Π¹¹ norm on \( R \) (cf. [3]) and put \( \varphi(i, x) = \omega \) if \( \neg R(i, x) \). Let

\[
 x \in B_i \iff R(i, x) \land \exists j < k(\varphi(i, x) < \varphi(j, x) \lor (\varphi(i, x) = \varphi(j, x) \land i < j)).
\]

Then \( B_i \) is Π¹¹. Note that if \( \exists i < k(x \notin A_i) \) then \( \forall i < k(x \in A_i \iff x \in B_i) \), while if \( \forall i < k(x \in A_i) \) then \( x \notin B_i \) for exactly one \( i < k \). Thus the \( B_i \) are as desired. □

Dellacherie asked whether Theorem 1 holds for countable families. A. S. Kechris and the author found a counterexample, which L. Harrington simplified considerably to the following. (Apparently, Kunen had earlier found a counterexample using forcing.)

COUNTEREXAMPLE 1. Let

\[
x \in A_0 \iff \text{codes an } \omega\text{-sequence } \{C_n\mid n < \omega\} \text{ of } \Pi_1^1 \text{ subsets of } \omega;
\]

\[
x \in A_{n+1} \iff x A_0 \land x \in C_n.
\]

Suppose \( \{B_n\mid n < \omega\} \) were an \( \omega\)-sequence of Π¹¹ sets reducing \( \{A_n\mid n < \omega\} \) in the sense of Theorem 1. Let \( x \) code \( \{B_n\mid n < \omega\} \); i.e. let \( B_n = C_n \) for \( n < \omega \). By (iii) let \( n \) be least so that \( x \notin B_n \). Then \( x \in A_m \) for \( m < n \) by the definition of \( \{A_n\mid n < \omega\} \). So \( x \in B_{n+1} \) by (ii). But then \( x \in A_{n+1} \), so \( x \in C_n = B_n \), a contradiction. □
If \( R \subseteq A \times B \) and \( x \in A \), we let \( R_x = \{ y \in B | (x, y) \in R \} \). \( R_x \) is called a section of \( R \).

**Theorem 2 (Dellacherie [1]).** Let \( R \subseteq \omega \times \omega \) be \( \Sigma_1^1 \), and \( R_x \) compact for each \( x \in \omega \). Then there is a \( \Sigma_1^1 \) set \( S \subseteq \omega \times \omega \) so that

(a) \( \forall x (S_x \text{ is compact and nonempty}) \); 

(b) \( \forall x (R_x \neq \emptyset \Rightarrow R_x = S_x) \).

**Proof.** Let \( R \) be as above. Define \( R^* \) by

\[ R^*(x, s) \iff s \in \omega^{<\omega} \land \exists y (s \subseteq y \land R(x, y)). \]

Then \( R^* \) is \( \Sigma_1^1 \), and \( R_x^* \) is a tree on \( \omega \) for each \( x \). Since \( R_x \) is closed, \( R_x \) is just the set of branches of \( R_x^* \); since \( R_x \) is compact, \( R_x^* \) is finitely branching. An easy boundedness argument yields a \( \Delta_1^1 \) set \( T^* \supseteq R^* \) so that \( T_x^* \) is an infinite, finitely branching tree on \( \omega \) for each \( x \).

Let \( \varphi \) be a \( \Pi_1^1 \) norm on \( (\omega \times \omega^{<\omega}) - R^* \) and define

\[ \psi(x, s) = \min \{ \varphi(x, t) | t \subseteq s \}. \]

Then \( \psi \) is a \( \Pi_1^1 \) norm on \( (\omega \times \omega^{<\omega}) - R^* \), and \( \{ s | \psi(x, s) > \alpha \} \) is a tree for each \( x, \alpha \). Let

\[ S^*(x, s) \iff T^*(x, s) \land (R^*(x, s) \lor T_x^* - \{ t | \psi(x, t) < \psi(x, s) \}) \text{ is finite}. \]

Then \( S^* \) is a finitely branching tree for each \( x \), and \( S^*_x = R_x^* \) if \( R_x^* \) is finite (i.e., if \( R_x \neq \emptyset \)). If \( R_x^* \) is finite, let \( \alpha \) be least so that \( T_x^* - \{ t | \psi(x, t) < \alpha \} \) is finite. Since \( T_x^* \) is finitely branching, \( \alpha = \beta + 1 \) for some \( \beta \). But then \( \psi(x, s) = \beta \) for infinitely many \( s \), and \( \psi(x, s) = \beta \Rightarrow S^*(x, s) \) by the definition of \( S^* \). Thus \( S^*_x \) is an infinite finitely branching tree for each \( x \). Let \( S(x, y) = \varphi \) be a branch of \( S^*_x \).

Dellacherie asked whether Theorem 2 holds with "\( \sigma \)-compact" replacing "compact".

**Counterexample 2.** Let \( \{ A_n | n < \omega \} \) be as defined in Counterexample 1. Let

\[ R(x, y) \iff \exists n (\forall i (y(i) = n) \land x \notin A_n). \]

Then \( R \) is \( \Sigma_1^1 \) and \( R_x \) is countable, hence \( \sigma \)-compact, for each \( x \). The argument of Counterexample 1 shows that if \( S \subseteq \omega \times \omega \) is \( \Sigma_1^1 \) and \( \forall x (R_x \neq \emptyset \Rightarrow R_x = S_x) \), then \( \exists x (R_x = \emptyset \lor S_x = \emptyset) \). □

The following is an old theorem.

**Theorem 3 (Lusin [2]).** Let \( R \subseteq \omega \times \omega \) be \( \Sigma_1^1 \) and \( R_x \) countable for each \( x \). Then there are \( \Sigma_1^1 \) sets \( S_n, n < \omega \), so that

(i) \( R = \bigcup_n S_n \);

(ii) for each \( x, n \), \( (S_n)_x \) has \( \leq 1 \) element.

Dellacherie asked whether Theorem 3 holds with "\( \sigma \)-compact" replacing "countable" and "is compact" replacing "has \( < 1 \) element".

**Counterexample 3.** For \( x \in \omega \), let \( W^x = \{ e \in \omega | e \) is the Gödel number of a well-order of \( \omega \) recursive in \( x \} \). The relation "\( y = W^x \)" is \( \Pi_1^1 \), as is "\( y = W^{W^x} \). Define now

\[ R(x, y) \iff y \in \omega^2 \land y \neq W^{W^x}. \]
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Then \( R \) is \( \Sigma^1_1 \), and \( R_x \) is \( \sigma \)-compact (in fact, open relative to \( \omega^2 \)) for each \( x \). Suppose \( \{ S_n \mid n < \omega \} \) were a family of \( \Sigma^1_1 \) sets as in the proposed extension of Theorem 3. Say that \( \langle x \in S_n \rangle \) is a \( \Sigma^1_1 \) relation of \( x, n \). Let \( S^*_n = \{ s \in \omega^\omega \mid \exists y (s \subseteq y \wedge y \in S_n) \} \). The sequence \( \langle S^*_n \mid n < \omega \rangle \) is recursive in \( W^z \). But \( W^\omega \) is the unique \( y \in \omega^2 \) so that \( y \) is not a branch of any of the trees \( S^*_n \). Thus \( W^\omega \) is \( \Delta^1_1 \) in \( W^z \), a contradiction. \( \square \)

Counterexample 3 raises the following question: let \( R \subseteq \omega \times \omega \) be \( \Sigma^1_1 \) with \( \sigma \)-compact sections. Must there be \( S \subseteq R, \Sigma^1_1 \) with compact sections, so that \( R_x \neq \emptyset \Rightarrow S_x \neq \emptyset \)? Clearly this is one instance of a family of problems generalizing the uniformization problem. If \( R, S \subseteq \omega \times \omega \) we say \( S \) almost uniformizes \( R \) iff \( S \subseteq R \) and \( \forall x (R_x \neq \emptyset \Rightarrow S_x \neq \emptyset) \). Notice that if \( R \) is \( \Sigma^1_1 \), then there is a \( \Sigma^1_1 \) with Borel sections which almost uniformizes \( R \); let \( S_{xy} \leftrightarrow (R_{xy} \wedge \omega^x_{y} = \omega^x_{y}' \wedge \omega_{y}' \) is the least ordinal not recursive in \( z \). On the other hand, L. Harrington has found a \( \Sigma^1_1 \) relation \( R \) such that, for every countable \( \alpha, R \) cannot be almost uniformized by a \( \Sigma^1_1 \) relation with \( \Pi^0_\alpha \) sections. We sketch his example for those familiar with [4]. Let \( P \) be the forcing notion of \( \S2 \) of [4]. Let \( R(x, T) \) iff \( T \) is \( P \)-generic over \( L_{\omega^1_1}[x] \) with respect to \( \Sigma^1_1 \) sentences. Then \( R \) is \( \Sigma^1_1 \), since \( R(x, T) \) iff \( \omega_{y}^x = \omega_{y'}^x \) and \( T \) is \( P \)-generic over \( L_{\omega^1_1}[x] \) with respect to ranked sentences. Lemma 1 of [4] implies that \( R \) cannot be almost uniformized by a \( \Sigma^1_1 \) relation with sections of bounded Borel class.

REFERENCES


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