PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 4, December 1980

PREDICTION n UNITS OF TIME AHEAD[1]

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ABSTRACT. The purpose of this note is to give a simple expression in terms of w of the quantities

$$\rho_n(w) = \inf_f \int_0^{2\pi} |1 + e^{in\theta} f|^2 w \, d\theta/2\pi \qquad (n = 0, 1, 2, \ldots),$$

where f ranges over the analytic trigonometric polynomials with mean value zero and w is nonnegative and summable on the circle.

Helson [2, pp. 21–22] said that it is unreasonable to expect to have a simple expression in terms of w for the quantities ρ_n except $n = 0$. $\rho_0(w) = \exp \int_0^{2\pi} \log w \, d\theta/2\pi$ is the famous Szegö theorem. We may assume $\log w$ is summable, because otherwise $\rho_n(w) = 0$ for all n.

THEOREM. *Let w be nonnegative and summable on the circle. Suppose* $\log w$ *is summable and*

$$\log w(\theta) \sim \sum_{j=-\infty}^{\infty} a_j e^{ij\theta}.$$

Then

$$\rho_n(w) = \inf_f \int_0^{2\pi} |1 + e^{in\theta} f|^2 w \, d\theta/2\pi$$
$$= e^{a_0} \times \sum_{j=0}^{n} \left| \sum{}' \frac{a_1^{m_1} a_2^{m_2} \cdots a_n^{m_n}}{m_1! m_2! \cdots m_n!} \right|^2$$

where $n \geq 1$ and f ranges over the analytic trigonometric polynomials with mean value zero and $\sum'$ is the summation of all permutations of nonnegative integers $m_1, m_2, \ldots, m_n$ with $m_1 + 2m_2 + \cdots + nm_n = j$ for each j.

PROOF. Set $g_1 = \sum_{l=1}^{n} a_l z^l$ and $g_2 = a_0/2 + \sum_{l \geq n+1}^{\infty} a_n z^n$, then their radial limits satisfy $w(\theta) = |\exp g_1(e^{i\theta})|^2 |\exp g_2(e^{i\theta})|^2$ a.e. θ. $\exp(g_1 + g_2)$ is an outer function [3, p. 61] and so $\exp g_2$ is outer. Hence, if we note that there exist positive numbers ε and M with $0 < \varepsilon \leq |\exp g_1(e^{i\theta})|^2 \leq M < \infty$, as in the proof of Szegö's theorem

Received by the editors July 9, 1979 and, in revised form, September 11, 1979 and November 9, 1979.
AMS (MOS) subject classifications (1970). Primary 60G25, 30A98, 46J15.
Key words and phrases. Prediction theory, Szegö's theorem, n units of time ahead, outer function, Fourier coefficients.
[1]Research supported in part from Grant-in-Aid Encouragement of Young Scientists.

$$\begin{aligned}\rho_n(w) &= \inf_f \int |1 + e^{in\theta}f|^2 |\exp g_2|^2 |\exp g_1|^2 \, d\theta/2\pi \\ &= \inf_f \int |\exp g_2 + e^{in\theta} f \exp g_2|^2 |\exp g_1|^2 \, d\theta/2\pi \\ &= \inf_f \int |e^{a_0/2} + e^{in\theta}f|^2 |\exp g_1|^2 \, d\theta/2\pi \\ &= e^{a_0} \inf_f \int |1 + e^{in\theta}f|^2 |\exp g_1|^2 \, d\theta/2\pi.\end{aligned}$$

Since $\exp g_1$ is an outer function, if the Fourier coefficients of $\exp g_1$ are $\{b_j\}$, then (cf. [1, pp. 184–187], [2, p. 22])

$$\begin{aligned}&\inf_f \int |1 + e^{in\theta}f|^2 |\exp g_1|^2 \, d\theta/2\pi \\ &\qquad = \sum_{j=0}^{n} |b_j|^2 \exp g_1(z) = \prod_{l=1}^{n} \exp(a_l z^l) = \sum \frac{(a_1 z)^{m_1} \cdots (a_n z^n)^{m_n}}{m_1! \cdots m_n!}\end{aligned}$$

where the m_j range independently over nonnegative integers. This implies the theorem.

The theorem shows the following:

$$\begin{aligned}\rho_1(w) &= \inf_f \int_0^{2\pi} |1 + e^{i\theta}f|^2 w \, d\theta/2\pi \\ &= \exp \int_0^2 \log w \, d\theta/2\pi \left\{ 1 + \left| \int_0^{2\pi} (\log w) e^{i\theta} \, d\theta/2\pi \right|^2 \right\}.\end{aligned}$$

If w is nonnegative and summable, and $\log w$ is summable, it is known (cf. [2, p. 20]) that $w = |g|^2$ for some outer g. The theorem gives a simple expression in terms of w of the Fourier coefficients of outer function g (cf. [4]).

We wish to express our appreciation to the referee for several valuable suggestions.

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