LINEAR VECTOR FIELDS ON $\tilde{G}_k(R^n)$

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Abstract. We determine the maximal number of linearly independent vector fields on the grassmannian of oriented $k$-subspaces of $R^n$, which are induced by linear transformations of $\Lambda^k(R^n)$.

Introduction. It was proved by Radon [5], Hurwitz [2] and Adams [1] that $\sigma_n = \text{span } S^{n-1}$ can be obtained as the maximal number of linear transformations of $R^n$ which induce linearly independent vector fields on $S^{n-1} \hookrightarrow R^n$.

In this article we consider the analogous problem of studying which linear transformations of $\Lambda^k(R^n)$ induce vector fields on $\tilde{G}_k(R^n) \hookrightarrow \Lambda^k(R^n)$. If $n$ is even and $k$ is odd we prove in §2 that the maximal number of linearly independent vector fields obtained in this fashion is given by $\sigma_n$. It will follow that $\text{span } \tilde{G}_k(R^n) > \text{span } S^{n-1}$ and in §4 we provide examples where equality or strict inequality occurs.

1. Preliminaries. Let $G_k(R^n)$ be the grassmannian of $k$-dimensional subspaces of $R^n$ and $\tilde{G}_k(R^n)$ be its oriented double covering, that is, the set of oriented, $k$-dimensional subspaces of $R^n$.

Consider the imbedding $\tilde{G}_k(R^n) \hookrightarrow \Lambda^k(R^n)$ which maps a $k$-subspace with oriented basis $v_1, \ldots, v_k$ to $v_1 \wedge v_2 \wedge \cdots \wedge v_k/\|v_1 \wedge \cdots \wedge v_k\|$. Let $P$ be a point in $\tilde{G}_k(R^n)$, represented by $v_1 \wedge \cdots \wedge v_k$ where $v_1, \ldots, v_k$ is an oriented orthonormal basis of $P$ and let $P^\perp$ denote its orthogonal complement in $R^n$. If we consider the curve $\gamma(t) = \gamma_1(t) \wedge \cdots \wedge \gamma_k(t)$ where $\gamma_i(0) = v_i$ and $\langle \gamma_i(t), \gamma_j(t) \rangle = \delta_{ij}$ for all $t$ then

$$\gamma'(0) = \sum_{i=1}^{k} v_1 \wedge \cdots \wedge \gamma_i(0) \wedge \cdots \wedge v_k$$

is a tangent vector at $P$, that is, belongs to $T_p(\tilde{G}_k(R^n))$. Since the set

$$\left\{ \sum_{i=1}^{k} v_1 \wedge \cdots \wedge X_i \wedge \cdots \wedge v_k, X_i \in P^\perp \right\}$$

is a vector subspace of $\Lambda^k(R^n)$ of dimension $k \cdot (n-k)$ and $\tilde{G}_k(R^n)$ is a manifold of the same dimension, we obtain all of $T_p(\tilde{G}_k(R^n))$ by this process.

2. Linear span of $\tilde{G}_k(R^n)$. Let $M$ be a compact manifold imbedded in $R^N$. It is natural to consider linear transformations of $R^N$ which induce vector fields on $M$. We call these linear vector fields.
We define the linear span of $M \rightarrow R^n$ as the maximal number of linear vector fields which are independent at each point. We denote by $\sigma_L(M)$ and $\sigma(M)$ the linear span and the span of $M$, respectively.

We observe that for $k$ or $n-k$ even, the Euler characteristic of $\tilde{G}_k(R^n)$ is different from zero [7], yielding span zero.

**Theorem.** For $k$ odd and $n$ even, $\sigma_L(\tilde{G}_k(R^n)) = \sigma(S^{n-1})$.

We recall that (see [1], [2], [5])

$$\sigma_n = \sigma(S^{n-1}) = 2^c + 8d - 1$$

where $n = 2^c + 4d(2a + 1), 0 < c < 3, a, d > 0$.

Moreover, this number is obtained as the maximal number of orthogonal transformations $T_j$ of $R^n$ satisfying

$$T_i^2 = -\text{Id} \quad \text{and} \quad T_iT_j + T_jT_i = 0, \quad i \neq j. \tag{\ast}$$

Let $T$ be a linear transformation of $R^n$ and define $\tilde{T} : \Lambda^k(R^n) \rightarrow \Lambda^k(R^n)$ by

$$\tilde{T}(v_1 \wedge \cdots \wedge v_k) = \sum_{i=1}^{k} v_1 \wedge \cdots \wedge T(v_i) \wedge \cdots \wedge v_k.$$

The proof of Lemma 1 shows that $\tilde{T}$ induces a tangent vector field if and only if $T$ is skew-symmetric.

**Lemma 1.** For $k$ odd, $k < n$, the transformation $\tilde{T} : \Lambda^k(R^n) \rightarrow \Lambda^k(R^n)$ gives a linear vector field on $\tilde{G}_k(R^n)$ without singularities (zeros) if and only if $T$ is a skew-symmetric isomorphism of $R^n$.

Now given a family $T_1, \ldots, T_m$ of orthogonal transformations satisfying (\ast), the linear vector fields $\tilde{T}_j, j = 1, \ldots, m$, are linearly independent at each point since

$$\sum_{j=1}^{m} a_j \tilde{T}_j = \left( \sum_{j=1}^{m} a_j T_j \right)^2$$

has a singularity if and only if

$$\det\left( \sum_{j=1}^{m} a_j T_j \right)^2 = \left( \sum_{j=1}^{m} a_j - a_j^2 \right)^n = 0.$$

Thus we have obtained $\sigma_m < \sigma_L(\tilde{G}_k(R^n))$. In order to prove equality we need:

**Lemma 2.** Any linear vector field on $\tilde{G}_k(R^n)$ is of the form

$$\tilde{T}(v_1 \wedge \cdots \wedge v_k) = \sum_{i=1}^{k} v_1 \wedge \cdots \wedge T(v_i) \wedge \cdots \wedge v_k$$

where $T$ is a skew-symmetric transformation of $R^n$.

Let $X_1, \ldots, X_s$ be linear vector fields on $\tilde{G}_k(R^n)$ and $T_1, \ldots, T_s$ the corresponding skew-symmetric transformations of $R^n$ (Lemma 2). Assume $X_1, \ldots, X_s$ are linearly independent at each point, i.e., every nontrivial combination has no singularities. It follows from Lemma 1 that every nontrivial combination of $T_1, \ldots, T_s$ is an isomorphism, and therefore (Lemma 1 again) they define linearly.
independent vector fields on \( \hat{G}_k(R^n) \approx S^{n-1} \). Thus \( \sigma_k(\hat{G}_k(R^n)) < \sigma_n \). To conclude the proof of the theorem it remains only to establish the lemmas.

**Remark.** The inequality \( \sigma(G_k(R^n)) > \sigma(S^{n-1}) \) is still true for \( k \) odd, \( n \) even since

\[
(\text{d}n)_p(\hat{T}(P)) = (\text{d}n)_{-p}(\hat{T}(-P))
\]

where \( \pi: \hat{G}_k(R^n) \to G_k(R^n) \) denotes the canonical projection and \(-P\) the subspace \( P \) with opposite orientation.

3. **Proof of the lemmas.**

**Proof of Lemma 1.** Assume \( T \) is a skew-symmetric isomorphism of \( R^n \). It is clear that \( \tilde{T}(v_1 \wedge \cdots \wedge v_k) \) is tangent to \( \hat{G}_k(R^n) \). Now if

\[
0 = \tilde{T}(v_1 \wedge \cdots \wedge v_k) = \sum_{i=1}^{k} v_1 \wedge \cdots \wedge T(v_i) \wedge \cdots \wedge v_k
\]

we have, for all \( j, 1 < j < k \),

\[
0 = \tilde{T}(v_1 \wedge \cdots \wedge v_k) \wedge v_j = v_1 \wedge \cdots \wedge T(v_j) \wedge \cdots \wedge v_k \wedge v_j
\]

which is impossible because a skew-symmetric isomorphism cannot have odd dimensional invariant subspaces (the restriction of \( T \) to \( \langle v_1, \ldots, v_k \rangle \) has an eigenvalue zero).

Suppose now that \( \tilde{T} \) is a linear vector field on \( \hat{G}_k(R^n) \). Since \( \hat{G}_k(R^n) \) is a submanifold of the standard sphere contained in \( \Lambda^k(R^n) \) we have, for every orthonormal set \( \{v_1, \ldots, v_k\} \),

\[
0 = \langle \tilde{T}(v_1 \wedge \cdots \wedge v_k), v_1 \wedge \cdots \wedge v_k \rangle = \sum_{i=1}^{k} \langle Tv_i, v_i \rangle = k\langle Tv_1, v_1 \rangle
\]

(note \( k < n \)) and therefore \( \tilde{T} \) is skew-symmetric.

If \( T \) is not an isomorphism pick \( \omega \in \ker T, ||\omega|| = 1 \), and \( v_1, \ldots, v_{k-1} \) in \( \omega^\perp \) such that the \((k-1)\)-subspace spanned by these vectors is invariant by \( T \) ("diagonalize" \( T \) over \( C \)). Then \( \tilde{T} \) has a singularity at the point \( v_1 \wedge \cdots \wedge v_{k-1} \wedge \omega \).

**Proof of Lemma 2.** Let \( H \) be a linear transformation of \( \Lambda^k(R^n) \) such that for every \( P \in \hat{G}_k(R^n) \), \( H(P) \in T_p(\hat{G}_k(R^n)) \). That is, if \( P \) is represented by \( v_1 \wedge \cdots \wedge v_k \) then

\[
H(P) = \sum_{i=1}^{k} v_1 \wedge \cdots \wedge H_p^i \wedge \cdots \wedge v_k \quad \text{where} \quad \langle H^i_p, v_i \rangle = 0.
\]

We will exhibit a skew-symmetric transformation \( T \) of \( R^n \) such that \( H = \tilde{T} \).

Let us define \( T \) on the canonical basis \( \{e_1, \ldots, e_n\} \) of \( R^n \) by

\[
\langle T(e_i), e_i \rangle = 0, \quad \langle T(e_i), e_j \rangle = \langle H_p^i, e_j \rangle, \quad j \neq i,
\]

where \( P \) is any \( k \)-subspace such that \( e_i \in P \) and \( e_j \in P^\perp \).

Assuming for a moment that \( T \) is well defined, we will verify that \( \tilde{T} = H \) on the canonical basis of \( \Lambda^k(R^n) \).
Let \( \mathcal{T} = e_{i_1} \wedge \cdots \wedge e_{k} \) so
\[
\mathcal{T}(\mathcal{Q}) = \sum_{r=1}^{k} e_{i_r} \wedge \cdots \wedge \left( \sum_{e_j \in \mathcal{Q}} \langle T(e_j), e_j \rangle e_j \right) \wedge \cdots \wedge e_{k} = H(\mathcal{Q}).
\]

Since \( \mathcal{T} \) gives a tangent vector field, \( T \) must be skew-symmetric.

We show now that \( T \) is well defined. Let \( P \) and \( R \) be two \( k \)-subspaces of \( \mathbb{R}^n \) such that \( e_i \in P \cap R \) and \( e_j \in P^\perp \cap R^\perp \). By induction on \( k - \text{dim}(P \cap R) \) we will prove that
\[
\langle H^i_P, e_j \rangle = \langle H^i_R, e_j \rangle. \tag{**}
\]

If \( \text{dim}(P \cap R) = k - 1 \) then
\[
P = (P \cap R) \oplus \langle \upsilon \rangle \quad \text{and} \quad R = (P \cap R) \oplus \langle \omega \rangle
\]
where \( \upsilon, \omega \in (P \cap R)^\perp \), \( ||\upsilon|| = ||\omega|| = 1 \). Let \( \{e_i, \upsilon_2, \ldots, \upsilon_{k-1}\} \) be an orthonormal basis of \( P \cap R \), so \( P = e_i \wedge \upsilon_2 \wedge \cdots \wedge \upsilon_{k-1} \wedge \upsilon \), \( R = e_i \wedge \upsilon_2 \wedge \cdots \wedge \upsilon_{k-1} \wedge \omega \), and consider the following two elements of \( \mathcal{G}_k(\mathbb{R}^n) \):
\[
Q = e_i \wedge \upsilon_2 \wedge \cdots \wedge \upsilon_{k-1} \wedge (\upsilon + \omega/\sqrt{2}),
\]
\[
S = e_i \wedge \upsilon_2 \wedge \cdots \wedge \upsilon_{k-1} \wedge (\upsilon - \omega/\sqrt{2}).
\]

Since \( H \) restricts to a vector field on \( \mathcal{G}_k(\mathbb{R}^n) \) we get
\[
H(P) = H^i_P \wedge \upsilon_2 \wedge \cdots \wedge \upsilon_{k-1} \wedge \upsilon + e_i \wedge P_1,
\]
\[
H(R) = H^i_R \wedge \upsilon_2 \wedge \cdots \wedge \upsilon_{k-1} \wedge \omega + e_i \wedge R_1,
\]
\[
H(Q) = H^i_Q \wedge \upsilon_2 \wedge \cdots \wedge \upsilon_{k-1} \wedge (\upsilon + \omega/\sqrt{2}) + e_i \wedge Q_1.
\]

It is not hard to check that
\[
\langle H(P), S \rangle = \frac{1}{\sqrt{2}} \langle H^i_P, e_j \rangle (1 - \langle \upsilon, \omega \rangle),
\]
\[
\langle H(R), S \rangle = \frac{1}{\sqrt{2}} \langle H^i_R, e_j \rangle (\langle \upsilon, \omega \rangle - 1)
\]
and
\[
\langle H(Q), S \rangle = 0.
\]

But \( H \) is a linear transformation on \( \Lambda^k(\mathbb{R}^n) \) and \( Q = (P + R)/\sqrt{2} \) so \( \langle H(P), S \rangle = -\langle H(R), S \rangle \) which implies
\[
\langle H^i_P, e_j \rangle = \langle H^i_R, e_j \rangle.
\]

If \( \text{dim}(P \cap R) = k - (s + 1) \) then \( P = (P \cap R) \oplus \langle \upsilon_1, \ldots, \upsilon_{s+1} \rangle \) and \( R = (P \cap R) \oplus \langle \omega_1, \ldots, \omega_{s+1} \rangle \) where \( \{\upsilon_1\}^{s+1}_{i=1} \) and \( \{\omega_1\}^{s+1}_{i=1} \) are orthonormal basis of \( (P \cap R)^\perp \) in \( P \) and \( R \), respectively. Consider \( V = (P \cap R) \oplus \langle \upsilon_1, \ldots, \upsilon_s, \omega_1 \rangle \). By inductive hypothesis
\[
\langle H^i_P, e_j \rangle = \langle H^i_V, e_j \rangle = \langle H^i_R, e_j \rangle. \quad \square
\]
4. Span and linear span. The theorem of §2 states that

$$\sigma(\tilde{G}_k(R^n)) > \sigma_L(\tilde{G}_k(R^n)).$$

In contrast with the sphere case \((k = 1)\) where \(\sigma(S^{n-1}) = \sigma_L(S^{n-1})\) we will exhibit an infinite number of examples where strict inequality occurs.

We recall that, for a simply connected manifold \(M\) of dimension \(4p + 1\), \(\sigma(M) > 1\), if and only if the index

$$I = \sum_{i=0}^{2p} b_{2i} \pmod{2}$$

is zero (see [6]). Here \(b_j\) stands for \(\dim H_j(M; R)\).

If \(p(z) = \sum b_j z^j\) denotes the Poincaré polynomial of \(M\) then

$$I = \frac{1}{2} \left( p(1) + p(-1) \right) = \frac{1}{2} \left( p(1) + \chi(M) \right) = \frac{1}{2} p(1).$$

From now on we will consider \(\tilde{G}_k(R^n)\), \(k = 2r + 1\) and \(n - k = 2s + 1\), where \(r\) and \(s\) have the same parity. These manifolds have dimension \(4p + 1\) and linear span equal to one.

The formula of the Poincaré polynomials for grassmannians is obtained in [4] by developing previous work of [3].

In our case

$$p(z) = (1 + z^{n-1}) \cdot \frac{(z^{4r+4} - 1) \cdots (z^{4r+4s} - 1)}{(z^4 - 1) \cdots (z^{4s} - 1)}.$$ 

Dividing each factor of the form \(z^j - 1\) by \(z - 1\) and taking the limit as \(z\) approaches one, we get

$$p(1) = 2 \binom{r+s}{s}$$

yielding that

$$I = 0 \quad \text{if and only if} \quad \binom{r+s}{s} \text{ is even.}$$

Therefore for the family considered above we obtain that

$$\sigma > \sigma_L \quad \text{if} \quad \binom{r+s}{s} \text{ is even and}$$

$$\sigma = \sigma_L \quad \text{if} \quad \binom{r+s}{s} \text{ is odd.}$$

REFERENCES


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