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LINEAR VECTOR FIELDS ON $\tilde{G}_k(R^n)$

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ABSTRACT. We determine the maximal number of linearly independent vector fields on the grassmannian of oriented k-subspaces of R^n, which are induced by linear transformations of $\Lambda^k(R^n)$.

Introduction. It was proved by Radon [5], Hurwitz [2] and Adams [1] that $\sigma_n = \operatorname{span} S^{n-1}$ can be obtained as the maximal number of linear transformations of R^n which induce linearly independent vector fields on $S^{n-1} \hookrightarrow R^n$.

In this article we consider the analogous problem of studying which linear transformations of $\Lambda^k(R^n)$ induce vector fields on $\tilde{G}_k(R^n) \hookrightarrow \Lambda^k(R^n)$. If n is even and k is odd we prove in §2 that the maximal number of linearly independent vector fields obtained in this fashion is given by σ_n. It will follow that $\operatorname{span} \tilde{G}_k(R^n) > \operatorname{span} S^{n-1}$ and in §4 we provide examples where equality or strict inequality occurs.

1. Preliminaries. Let $G_k(R^n)$ be the grassmannian of k-dimensional subspaces of R^n and $\tilde{G}_k(R^n)$ be its oriented double covering, that is, the set of oriented, k-dimensional subspaces of R^n.

Consider the imbedding $\tilde{G}_k(R^n) \hookrightarrow \Lambda^k(R^n)$ which maps a k-subspace with oriented basis $v_1, \ldots, v_k$ to $v_1 \wedge v_2 \wedge \cdots \wedge v_k / \|v_1 \wedge \cdots \wedge v_k\|$. Let P be a point in $\tilde{G}_k(R^n)$, represented by $v_1 \wedge \cdots \wedge v_k$ where $v_1, \ldots, v_k$ is an oriented orthonormal basis of P and let $P^\perp$ denote its orthogonal complement in R^n. If we consider the curve $\gamma(t) = \gamma_1(t) \wedge \cdots \wedge \gamma_k(t)$ where $\gamma_i(0) = v_i$ and $\langle \gamma_i(t), \gamma_j(t) \rangle = \delta_{ij}$ for all t then

$$\dot{\gamma}(0) = \sum_{i=1}^{k} v_1 \wedge \cdots \wedge \dot{\gamma}_i(0) \wedge \cdots \wedge v_k$$

is a tangent vector at P, that is, belongs to $T_p(\tilde{G}_k(R^n))$. Since the set

$$\left\{ \sum_{i=1}^{k} v_1 \wedge \cdots \wedge X_i \wedge \cdots \wedge v_k, X_i \in P^\perp \right\}$$

is a vector subspace of $\Lambda^k(R^n)$ of dimension $k \cdot (n-k)$ and $\tilde{G}_k(R^n)$ is a manifold of the same dimension, we obtain all of $T_p(\tilde{G}_k(R^n))$ by this process.

2. Linear span of $\tilde{G}_k(R^n)$. Let M be a compact manifold imbedded in R^N. It is natural to consider linear transformations of R^N which induce vector fields on M. We call these *linear vector fields*.

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We define the *linear span* of $M \hookrightarrow R^N$ as the maximal number of linear vector fields which are independent at each point. We denote by $\sigma_L(M)$ and $\sigma(M)$ the linear span and the span of M, respectively.

We observe that for k or $n-k$ even, the Euler characteristic of $\tilde{G}_k(R^n)$ is different from zero [7], yielding span zero.

THEOREM. *For k odd and n even, $\sigma_L(\tilde{G}_k(R^n)) = \sigma(S^{n-1})$.*

We recall that (see [1], [2], [5])

$$\sigma_n = \sigma(S^{n-1}) = 2^c + 8d - 1$$

where $n = 2^{c+4d}(2a+1)$, $0 \leq c \leq 3$, $a, d \geq 0$.

Moreover, this number is obtained as the maximal number of orthogonal transformations T_i of R^n satisfying

$$T_i^2 = -\mathrm{Id} \quad \text{and} \quad T_iT_j + T_jT_i = 0, \qquad i \neq j. \tag{$*$}$$

Let T be a linear transformation of R^n and define $\tilde{T}\colon \Lambda^k(R^n) \to \Lambda^k(R^n)$ by

$$\tilde{T}(v_1 \wedge \cdots \wedge v_k) = \sum_{i=1}^{k} v_1 \wedge \cdots \wedge T(v_i) \wedge \cdots \wedge v_k.$$

The proof of Lemma 1 shows that $\tilde{T}$ induces a tangent vector field if and only if T is skew-symmetric.

LEMMA 1. *For k odd, $k < n$, the transformation $\tilde{T}\colon \Lambda^k(R^n) \to \Lambda^k(R^n)$ gives a linear vector field on $\tilde{G}_k(R^n)$ without singularities (zeros) if and only if T is a skew-symmetric isomorphism of R^n.*

Now given a family $T_1, \ldots, T_{\sigma_n}$ of orthogonal transformations satisfying ($*$), the linear vector fields $\tilde{T}_j$, $j = 1, \ldots, \sigma_n$, are linearly independent at each point since

$$\sum_{j=1}^{\sigma_n} a_j \tilde{T}_j = \left(\sum_{j=1}^{\sigma_n} a_j T_j\right)^{\sim}$$

has a singularity if and only if

$$\det\left(\sum_{j=1}^{\sigma_n} a_j T_j\right)^2 = \left(\sum_{j=1}^{\sigma_n} - a_j^2\right)^n = 0.$$

Thus we have obtained $\sigma_n \leq \sigma_L(\tilde{G}_k(R^n))$. In order to prove equality we need:

LEMMA 2. *Any linear vector field on $\tilde{G}_k(R^n)$ is of the form*

$$\tilde{T}(v_1 \wedge \cdots \wedge v_k) = \sum_{i=1}^{k} v_1 \wedge \cdots \wedge T(v_i) \wedge \cdots \wedge v_k$$

where T is a skew-symmetric transformation of R^n.

Let $X_1, \ldots, X_s$ be linear vector fields on $\tilde{G}_k(R^n)$ and $T_1, \ldots, T_s$ the corresponding skew-symmetric transformations of R^n (Lemma 2). Assume $X_1, \ldots, X_s$ are linearly independent at each point, i.e., every nontrivial combination has no singularities. It follows from Lemma 1 that every nontrivial combination of $T_1, \ldots, T_s$ is an isomorphism, and therefore (Lemma 1 again) they define linearly

independent vector fields on $\tilde{G}_1(R^n) \approx S^{n-1}$. Thus $\sigma_L(\tilde{G}_k(R^n)) \leq \sigma_n$. To conclude the proof of the theorem it remains only to establish the lemmas.

REMARK. The inequality $\sigma(G_k(R^n)) \geq \sigma(S^{n-1})$ is still true for k odd, n even since

$$(d\pi)_P(\tilde{T}(P)) = (d\pi)_{-P}(\tilde{T}(-P))$$

where $\pi\colon \tilde{G}_k(R^n) \to G_k(R^n)$ denotes the canonical projection and $-P$ the subspace P with opposite orientation.

3. Proof of the lemmas.

PROOF OF LEMMA 1. Assume T is a skew-symmetric isomorphism of R^n. It is clear that $\tilde{T}(v_1 \wedge \cdots \wedge v_k)$ is tangent to $\tilde{G}_k(R^n)$. Now if

$$0 = \tilde{T}(v_1 \wedge \cdots \wedge v_k) = \sum_{i=1}^{k} v_1 \wedge \cdots \wedge T(v_i) \wedge \cdots \wedge v_k$$

we have, for all j, $1 \leq j \leq k$,

$$0 = \tilde{T}(v_1 \wedge \cdots \wedge v_k) \wedge v_j = v_1 \wedge \cdots \wedge T(v_j) \wedge \cdots \wedge v_k \wedge v_j$$

which is impossible because a skew-symmetric isomorphism cannot have odd dimensional invariant subspaces (the restriction of T to $\langle v_1, \ldots, v_k\rangle$ has an eigenvalue zero).

Suppose now that $\tilde{T}$ is a linear vector field on $\tilde{G}_k(R^n)$. Since $\tilde{G}_k(R^n)$ is a submanifold of the standard sphere contained in $\Lambda^k(R^n)$ we have, for every orthonormal set $\{v_1, \ldots, v_k\}$,

$$0 = \langle \tilde{T}(v_1 \wedge \cdots \wedge v_k), v_1 \wedge \cdots \wedge v_k\rangle = \sum_{i=1}^{k} \langle Tv_i, v_i\rangle = k\langle Tv_1, v_1\rangle$$

(note $k < n$) and therefore T is skew-symmetric.

If T is not an isomorphism pick $\omega \in \ker T$, $\|\omega\| = 1$, and $v_1, \ldots, v_{k-1}$ in $\omega^\perp$ such that the $(k-1)$-subspace spanned by these vectors is invariant by T ("diagonalize" T over $\mathbf{C}$). Then $\tilde{T}$ has a singularity at the point $v_1 \wedge \cdots \wedge v_{k-1} \wedge \omega$. $\square$

PROOF OF LEMMA 2. Let H be a linear transformation of $\Lambda^k(R^n)$ such that for every $P \in \tilde{G}_k(R^n)$, $H(P) \in T_P(\tilde{G}_k(R^n))$. That is, if P is represented by $v_1 \wedge \cdots \wedge v_k$ then

$$H(P) = \sum_{i=1}^{k} v_1 \wedge \cdots \dot{\wedge} H_P^i \wedge \cdots \wedge v_k \quad \text{where } \langle H_P^i, v_i\rangle = 0.$$

We will exhibit a skew-symmetric transformation T of R^n such that $H = \tilde{T}$.

Let us define T on the canonical basis $\{e_1, \ldots, e_n\}$ of R^n by

$$\langle T(e_i), e_i\rangle = 0, \quad \langle T(e_i), e_j\rangle = \langle H_P^i, e_j\rangle, \qquad j \neq i,$$

where P is any k-subspace such that $e_i \in P$ and $e_j \in P^\perp$.

Assuming for a moment that T is well defined, we will verify that $\tilde{T} = H$ on the canonical basis of $\Lambda^k(R^n)$.

Let $Q = e_{i_1} \wedge \cdots \wedge e_{i_k}$ so

$$\tilde{T}(Q) = \sum_{r=1}^{k} e_{i_1} \wedge \cdots \wedge \left(\sum_{e_j \in Q^\perp} \langle T(e_{i_r}), e_j \rangle e_j \right) \wedge \cdots \wedge e_{i_k}$$

$$= \sum_{r=1}^{k} e_{i_1} \wedge \cdots \wedge \left(\sum_{e_j \in Q^\perp} \langle H_Q^i, e_j \rangle e_j \right) \wedge \cdots \wedge e_{i_k} = H(Q).$$

Since $\tilde{T}$ gives a tangent vector field, T must be skew-symmetric.

We show now that T is well defined. Let P and R be two k-subspaces of R^n such that $e_i \in P \cap R$ and $e_j \in P^\perp \cap R^\perp$. By induction on $k - \dim(P \cap R)$ we will prove that

$$\langle H_P^i, e_j \rangle = \langle H_R^i, e_j \rangle. \tag{**}$$

If $\dim(P \cap R) = k - 1$ then

$$P = (P \cap R) \oplus \langle v \rangle \quad \text{and} \quad R = (P \cap R) \oplus \langle \omega \rangle$$

where $v, \omega \in (P \cap R)^\perp$, $\|v\| = \|\omega\| = 1$. Let $\{e_i, v_2, \ldots, v_{k-1}\}$ be an orthonormal basis of $R \cap P$, so $P = e_i \wedge v_2 \wedge \cdots \wedge v_{k-1} \wedge v$, $R = e_i \wedge v_2 \wedge \cdots \wedge v_{k-1} \wedge \omega$, and consider the following two elements of $\tilde{G}_k(R^n)$:

$$Q = e_i \wedge v_2 \wedge \cdots \wedge v_{k-1} \wedge (v + \omega/\sqrt{2}),$$

$$S = e_j \wedge v_2 \wedge \cdots \wedge v_{k-1} \wedge (v - \omega/\sqrt{2}).$$

Since H restricts to a vector field on $\tilde{G}_k(R^n)$ we get

$$H(P) = H_P^i \wedge v_2 \wedge \cdots \wedge v_{k-1} \wedge v + e_i \wedge P_1,$$

$$H(R) = H_R^i \wedge v_2 \wedge \cdots \wedge v_{k-1} \wedge \omega + e_i \wedge R_1,$$

$$H(Q) = H_Q^i \wedge v_2 \wedge \cdots \wedge v_{k-1} \wedge (v + \omega/\sqrt{2}) + e_i \wedge Q_1.$$

It is not hard to check that

$$\langle H(P), S \rangle = \frac{1}{\sqrt{2}} \langle H_P^i, e_j \rangle (1 - \langle v, \omega \rangle),$$

$$\langle H(R), S \rangle = \frac{1}{\sqrt{2}} \langle H_R^i, e_j \rangle (\langle v, \omega \rangle - 1)$$

and

$$\langle H(Q), S \rangle = 0.$$

But H is a linear transformation on $\Lambda^k(R^n)$ and $Q = (P + R)/\sqrt{2}$ so $\langle H(P), S \rangle = -\langle H(R), S \rangle$ which implies

$$\langle H_P^i, e_j \rangle = \langle H_R^i, e_j \rangle.$$

If $\dim(P \cap R) = k - (s + 1)$ then $P = (P \cap R) \oplus \langle v_1, \ldots, v_{s+1} \rangle$ and $R = (P \cap R) \oplus \langle \omega_1, \ldots, \omega_{s+1} \rangle$ where $\{v_i\}_{i=1}^{s+1}$ and $\{\omega_i\}_{i=1}^{s+1}$ are orthonormal basis of $(P \cap R)^\perp$ in P and R, respectively. Consider $V = (P \cap R) \oplus \langle v_1, \ldots, v_s, \omega_1 \rangle$. By inductive hypothesis

$$\langle H_P^i, e_j \rangle = \langle H_V^i, e_j \rangle = \langle H_R^i, e_j \rangle. \quad \square$$

4. Span and linear span. The theorem of §2 states that

$$\sigma(\tilde{G}_k(R^n)) \geqslant \sigma_n = \sigma_L(\tilde{G}_k(R^n)).$$

In contrast with the sphere case ($k = 1$) where $\sigma(S^{n-1}) = \sigma_L(S^{n-1})$ we will exhibit an infinite number of examples where strict inequality occurs.

We recall that, for a simply connected manifold M of dimension $4p + 1$, $\sigma(M) > 1$, if and only if the index

$$I = \sum_{i=0}^{2p} b_{2i} \pmod 2$$

is zero (see [6]). Here b_j stands for $\dim H_j(M; R)$.

If $p(z) = \sum b_j z^j$ denotes the Poincaré polynomial of M then

$$I = \tfrac{1}{2}(p(1) + p(-1)) = \tfrac{1}{2}(p(1) + \chi(M)) = \tfrac{1}{2}p(1).$$

From now on we will consider $\tilde{G}_k(R^n)$, $k = 2r + 1$ and $n - k = 2s + 1$, where r and s have the same parity. These manifolds have dimension $4p + 1$ and linear span equal to one.

The formula of the Poincaré polynomials for grassmannians is obtained in [4] by developing previous work of [3].

In our case

$$p(z) = (1 + z^{n-1}) \cdot \frac{(z^{4r+4} - 1) \cdots (z^{4r+4s} - 1)}{(z^4 - 1) \cdots (z^{4s} - 1)}.$$

Dividing each factor of the form $z^l - 1$ by $z - 1$ and taking the limit as z approaches one, we get

$$p(1) = 2\binom{r+s}{s}$$

yielding that

$$I = 0 \quad \text{if and only if } \binom{r+s}{s} \text{ is even.}$$

Therefore for the family considered above we obtain that

$$\sigma > \sigma_L \quad \text{if } \binom{r+s}{s} \text{ is even and}$$

$$\sigma = \sigma_L \quad \text{if } \binom{r+s}{s} \text{ is odd.}$$

References

1. F. Adams, *Vector fields on spheres*, Ann. of Math. (2) **75** (1962), 603–632.
2. A. Hurwitz, *Über die Komposition der quadratische Formen*, Math. Ann. **88** (1923), 1–25.
3. H. Iwamoto, *On integral invariants and Betti numbers of symmetric Riemannian manifolds*. I, J. Math. Soc. Japan **1** (1949), 91–110.
4. C. Rader, *The Betti numbers of Grassmannians*, Atas do XIII Colóquio Brasileiro de Matemática, Poços de Caldas, 1979.
5. J. Radon, *Lineare Scharen orthogonaler Matrizen*, Abh. Math. Sem. Univ. Hamburg **1** (1922), 1–14.
6. E. Thomas, *Vector fields on manifolds*, Bull. Amer. Math. Soc. **75** (1969), 643–683.
7. J. Wolf, *Spaces of constant curvature*, Publish or Perish, Boston, Mass., 1979.

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