S-SETS AND S-PERFECT MAPPINGS

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Abstract. In this note we generalize the notion of an S-closed space to S-sets of a space. Among our characterizations of S-sets, we show that a subset A of an extremally disconnected space X is an S-set in X if and only if A is an H-set in X. We also investigate conditions under which mappings or their inverses preserve S-sets.

1. Introduction. For convenience, all spaces are assumed to be Hausdorff although many of the assertions made below apply in a more general setting. Concerning notation, \( \bar{A} \) (\( A^\circ \)) will be used to denote the closure (interior) of A in a space \( X \), and when necessary, \( \bar{A}^\tau \) will denote the closure of \( A \) in the space \( X \) endowed with the topology \( \tau \). If \( x \in X \), \( \pi_x \) is the open neighborhood filter at \( x \) and \( \pi_x = \{ N \mid N \cap \pi_x \neq \emptyset \} \). Correspondingly, \( S_x \) (and, when necessary, \( S_x(X, \tau) \)) will denote the filter of all semi-open (abbreviated s.o.) subsets of \( X \) which contain \( x \). A subset \( S \) of \( X \) is semi-open if there exists an open subset \( O \) of \( X \) such that \( O \subseteq S \subseteq \bar{O} \). \( S_x \) is defined analogously and the family of all s.o. subsets of a space \( X \) is denoted \( S(X) \).

The concept of an H-set was introduced by N. Veličko in [9]. A subset \( A \) of a Hausdorff space \( X \) is an H-set if every cover of \( A \) by open subsets of \( X \) contains a finite dense subsystem, i.e. a finite subfamily whose closures in \( X \) cover \( A \). This concept was independently introduced in [6] and called H-closed relative to \( X \). A Hausdorff space \( X \) is H-closed if it is an H-set (relative to \( X \)). Veličko showed that \( \theta \)-closed subsets of H-closed spaces are H-sets (a subset \( A \) of \( X \) is \( \theta \)-closed if \( A = sp(A) \equiv \{ x \in X \mid N \cap A \neq \emptyset \text{ for every } N \in \pi_x \} \) and that H-sets of H-closed Urysohn spaces are \( \theta \)-closed.

More recently, T. Thompson defined a space \( X \) to be S-closed if every s.o. cover of \( X \) contains a finite dense subsystem. A subset \( A \) of a Hausdorff space \( X \) will be called an S-set (relative to \( X \)) if every cover of \( A \) by s.o. subsets of \( X \) contains a finite dense subsystem.

In §2, we present some characterizations of S-sets and obtain results analogous to those of Veličko mentioned above. We also define an S-point. This concept was introduced by E. K. van Douwen in [8] who noted that remote points of a completely regular space \( X \) are S-points of \( \beta X \). We will show that extremely
disconnected spaces consist entirely of \(S\)-points. (A space \(X\) is \textit{extremely disconnected}, abbreviated e.d., if regular closed subsets of \(X\) are open; a subset \(A\) of \(X\) is \textit{regular closed} if \(A = \overline{O}\) for some open subset \(O\) of \(X\).)

In §3, we introduce the definition of \(s\)-perfect mapping and we consider the preservation of \(S\)-sets by mappings and their inverses. We produce a large class of \(s\)-perfect mappings and exhibit a class of spaces which possess precisely the same \(S\)-sets. Through the use of \(S\)-points we obtain necessary conditions for a mapping to be \(s\)-perfect, and, as a result, obtain conditions under which the domain of an \(s\)-perfect mapping is e.d.

In §4, we present a characterization of compact e.d. spaces which enables us to observe that the class of \(s\)-perfect mappings contains a well-known subclass. We also note conditions under which the image of a compact e.d. space is necessarily e.d.

2. \(S\)-sets. For a subset \(A\) of \(X\), the \(s\)-closure of \(A\), denoted \(\text{cls}_s A\), is the set 
\[
\{ x \in X | \overline{S} \cap A \neq \emptyset \text{ for every } S \in S_x \}. 
\]
If \(A\) is a subset of \(X\) and \(\mathcal{F}\) is a filter base on \(X\) we say that \(\mathcal{F}\) meets \(A\) if \(F \cap A \neq \emptyset\) for every \(F \in \mathcal{F}\). \(\mathcal{F}\) is said to \(s\)-accumulate at \(x\) (\(x\) is an \(S\)-accumulation point of \(\mathcal{F}\)), written \(x \in \text{sad}_s \mathcal{F}\), if \(\mathcal{F}\) meets each \(\overline{S} \in S_x\), and \(\mathcal{F}\) \(s\)-converges to \(x\) if every \(\overline{S} \in S_x\) contains some \(F \in \mathcal{F}\). Correspondingly, we say that \(\mathcal{F}\) \(s_*\)-accumulates at \(x\), written \(x \in \text{sad}_* \mathcal{F}\), if \(\mathcal{F}\) meets each \(S \in S_x\).

A straightforward application of Zorn’s Lemma yields

\text{Lemma 2.1. Let } A \text{ be a nonempty subset of a space } X. \text{ If } \mathcal{F} \text{ is a filter base on } X \text{ which meets } A, \text{ then } \mathcal{F} \text{ is contained in a maximal filter base which also meets } A.\]

\text{Proposition 2.2. The following are equivalent for a space } X.

(\text{i}) \(A\) is an \(S\)-set.

(\text{ii}) Every maximal filter base on \(X\) which meets \(A\) \(s\)-converges to some point in \(A\).

(\text{iii}) Every filter base on \(X\) which meets \(A\) \(s\)-accumulates at some point in \(A\).

(\text{iv}) Every open filter base on \(X\) which meets \(A\) \(s_*\)-accumulates at some point in \(A\).

(\text{v}) If \(\mathcal{F} = \{ F_\alpha \}_{\alpha \in A} \) is an open filter base on \(X\) which meets \(A\), then \(\bigcap_{\alpha \in A} (\overline{F_\alpha})^o \cap A \neq \emptyset\).

\text{Proof. (i) } \Rightarrow (\text{ii}). \text{ Let } A \text{ be an } S\text{-set and suppose } \mathcal{U} \text{ is a maximal filter base on } X \text{ which meets } A \text{ and does not } s\text{-converge to some point in } A. \text{ Then, if } x \in A, \text{ there exists } S_x \in S_x \text{ such that } U \cap (X \setminus \overline{S_x}) \neq \emptyset \text{ for every } U \in \mathcal{U}. \text{ The maximality of } \mathcal{U} \text{ implies that } \mathcal{U} \supseteq \{ U \cap A | U \in \mathcal{U} \} \text{ and hence also that } \mathcal{U} \supseteq \{ U \cap (X \setminus \overline{S_x}) | U \in \mathcal{U} \}. \text{ Thus there exists } U_x \in \mathcal{U} \text{ such that } U_x \cap \overline{S_x} = \emptyset. \text{ } (\overline{S_x})_{x \in A} \text{ is an s.o. cover of } A, \text{ and } A \text{ is therefore contained in some } \bigcup_{\alpha = 1}^n \overline{S_{\alpha}}. \text{ Now } \bigcap_{\alpha = 1}^n U_x \in \mathcal{U} \text{ and } (\bigcap_{\alpha = 1}^n U_x) \cap A \subseteq (\bigcap_{\alpha = 1}^n U_x) \cap (\bigcup_{\alpha = 1}^n \overline{S_{\alpha}}) = \emptyset, \text{ a contradiction since } \mathcal{U} \text{ meets } A. \text{ This establishes (ii). Other implications are straightforward and therefore omitted.}\]

\text{Proposition 2.3. In an e.d. space every } H\text{-set is an } S\text{-set.}\]

\text{Proof. If } \{ S_{\alpha} \}_{\alpha \in A} \text{ is an s.o. cover of an } H\text{-set, } A, \text{ then } \{ \overline{S_{\alpha}} \}_{\alpha \in A} \text{ is an open cover of } A \text{ and some finite union must contain } A. \text{ Hence } A \text{ is an } S\text{-set.}
**Remark 2.4.** Since $S$-closed spaces are e.d. [7, Theorem 7], an $H$-closed space is e.d. if and only if every $H$-set is an $S$-set. However, $\beta N \setminus N$ is an $H$-set relative to the e.d. space $\beta N$ which is not e.d. [2, 6R(1)]. Thus $S$-sets need not be e.d.

**Corollary 2.5.** A subset $A$ of a $S$-closed space $X$ is an $S$-set if and only if $A = \text{cl}_S A$.

**Proof.** Note that $X$ is an e.d. Urysohn space so that $A$ is an $H$-set if and only if $A = \text{cl}_S A$ [9]. Thus by Proposition 2.3 $A$ is an $S$-set if and only if $A = \text{cl}_S A$.

A point $x \in X$ will be called an $S$-point (of $X$) if $x \in (\tilde{S})^o$ for every $S \in S_x$. $x \in X$ is called an *ordinary point* if $x$ is not an $S$-point.

**Remark 2.6.** It is easily seen that $x$ is an $S$-point of $X$ if and only if $x$ is e.d. at $x$ [8]. For if $x \not\in (\tilde{S})^o$ where $S \in S_x$, then $x \in (\tilde{S}^o) \cap (X \setminus S)$ so that $X$ is not e.d. at $x$, and if $x \in U \cap V$ where $U$, $V$ are disjoint and open in $X$, then $x \not\in (\tilde{U})^o$ so that $x$ is not an $S$-point of $X$. Thus a space $X$ is e.d. if and only if every $x \in X$ is an $S$-point of $X$. Note that for an arbitrary subset $A$ of $X$, $\text{cl}_S A \setminus \text{cl}_S A$ does not contain any $S$-points. Thus $\text{cl}_S A = \text{cl}_S A$ whenever $\text{cl}_S A \setminus A$ consists entirely of $S$-points as, for example, in e.d. spaces.

### 3. $S$-perfect mappings

A mapping $f: X \to Y$ is said to be *irresolute* [1] if given $S \in \mathcal{S}(Y)$, $f^{-1}(S) \in \mathcal{S}(X)$.

**Proposition 3.1.** If $f: X \to Y$ is a continuous irresolute mapping, then $f$ preserves $S$-sets.

**Proof.** Let $A$ be an $S$-set in $X$ and let $\{S_a\}_{a \in A}$ be an s.o. cover of $f(A)$. Then $\{f^{-1}(S_a)\}_{a \in A}$ is an s.o. cover of $A$ so that $A$ is contained in some $\bigcup_{i=1}^n f^{-1}(S_a)$.

Hence

$$f(A) \subseteq \bigcup_{i=1}^n f(f^{-1}(S_a)) \subseteq \bigcup_{i=1}^n S_a$$

which proves our assertion.

A mapping $f: X \to Y$ is called an *$s$-closed mapping* if $\text{cl}_S f(A) \subseteq f(\text{cl}_S A)$ for every subset $A$ of $X$, and $f$ is said to be *$s$-perfect* if $f$ is an $s$-closed mapping and point inverses are $S$-sets.

**Lemma 3.2.** A mapping $f: X \to Y$ is $s$-perfect if and only if $\text{sad} f(\mathcal{F}) \subseteq f(\text{sad} \mathcal{F})$ for every filter base $\mathcal{F}$ on $X$.

**Proof of necessity.** Let $\mathcal{F}$ be a filter base on $X$ and let $y \in Y \setminus f(\text{sad} \mathcal{F})$. For $x \in f^{-1}(y)$, there exist $S_x \in S_X$ and $F_x \in \mathcal{F}$ such that $S_x \cap F_x = \emptyset$. $\{S_x\}_{x \in f^{-1}(y)}$ is an s.o. cover of the $S$-set, $f^{-1}(y)$, so that $f^{-1}(y)$ is contained in some $\bigcup_{i=1}^n S_x$. $F = \bigcap_{i=1}^n F_x \in \mathcal{F}$ and $F \cap (\bigcup_{i=1}^n S_x) = \emptyset$. Hence $\text{cl}_S F \cap f^{-1}(y) = \emptyset$. Since $f$ is an $s$-closed mapping, $y \not\in \text{cl}_S f(F)$ and therefore, $y \not\in \text{sad} f(\mathcal{F})$.

**Proof of sufficiency.** Let $A$ be a subset of $X$ and let $y \in \text{cl}_S f(A)$. $\mathcal{F} = \{F \subseteq X|A \subseteq F\}$ is a filter base on $X$ such that $y \in \text{sad} f(\mathcal{F}) \subseteq f(\text{sad} \mathcal{F})$. Hence $\emptyset \neq \text{sad} \mathcal{F} \cap f^{-1}(y) \subseteq \text{cl}_S A \cap f^{-1}(y)$. This proves that $f$ is an $s$-closed mapping.

We shall use Proposition 2.2 to show that point inverses are $S$-sets. Let $\mathcal{F}$ be a
filter base on $X$ which meets $f^{-1}(y)$. Then $y \in f(F)$ for every $F \in \mathcal{F}$ so that $y \in \text{sad } f(\mathcal{F}) \subseteq f(\text{sad } \mathcal{F})$. Hence $\text{sad } \mathcal{F} \cap f^{-1}(y) \neq \emptyset$. By Proposition 2.2, $f^{-1}(y)$ is an $S$-set. This completes the proof.

**Proposition 3.3.** If $f: X \to Y$ is $s$-perfect, then inverse images of $S$-sets are $S$-sets.

**Proof.** We shall use Proposition 2.2. Let $A$ be an $S$-set in $Y$ and let $\mathcal{F}$ be a filter base on $X$ which meets $f^{-1}(A)$. Set $\mathcal{G} = \{F \cap f^{-1}(A) | F \in \mathcal{F}\}$. Then $f(\mathcal{G})$ is a filter base on $Y$ which meets $A$ and $f(\text{sad } \mathcal{G}) \cap A \supseteq f(\text{sad } \mathcal{G}) \cap A \neq \emptyset$ (Proposition 2.2 and Lemma 3.2). Thus $\text{sad } \mathcal{F} \cap f^{-1}(A) \supseteq \text{sad } \mathcal{G} \cap f^{-1}(A) \neq \emptyset$ so that $f^{-1}(A)$ is an $S$-set by Proposition 2.2.

**Proposition 3.4.** If $f: X \to Y$ is a continuous surjection and $X$ is compact and e.d., then $f$ is $s$-perfect.

**Proof.** Let $\mathcal{F}$ be a filter base on $X$ and let $y \in \text{sad } f(\mathcal{F})$. Then $\mathcal{G} = \{f^{-1}(\overline{N}) \cap F | N \in \eta_y, F \in \mathcal{F}\}$ is a filter base on $X$, and since $X$ is $S$-closed, $\mathcal{G}$ $s$-accumulates at some $x \in X$ [7, Theorem 2]. $x \in \text{sad } \mathcal{G} \subseteq \text{sad } \mathcal{F}$. We show that $x \in f^{-1}(y)$. Suppose $x \notin f^{-1}(y)$. Since $f$ is a surjection, $Y$ is compact and there exist $N_y \in \eta_y, \overline{N}_{f(x)} \in \eta_{f(x)}$ such that $\overline{N}_y \cap \overline{N}_{f(x)} = \emptyset$. The continuity of $f$ implies there exists $N_x \in \eta_x$ such that $f(N_x) \subseteq \overline{N}_{f(x)}$. But $x \in \text{sad } \mathcal{G}$ so that $\overline{N}_x \cap f^{-1}(\overline{N}_y) \neq \emptyset$. Therefore $f(\overline{N}_x) \cap \overline{N}_y \neq \emptyset$, a contradiction. We conclude that $x \in f^{-1}(y)$ so that $y \in f(\text{sad } \mathcal{F})$ and $f$ is $s$-perfect (Lemma 3.2).

**Corollary 3.5.** If $f: X \to Y$ is a continuous mapping of an $S$-closed space into a Urysohn space, then $f$ is $s$-perfect.

**Proof.** The proof is similar to the proof of Proposition 3.4.

Let $X$ be a fixed set and let $\mathcal{F}$ be the family of all topologies on $X$ having the same family of regular open subsets of $X$ (a subset $R$ of $X$ is regular open if $R = C^0$ for some regular closed subset $C$ of $X$). The elements of $\mathcal{F}$ are said to be r.o.-equivalent.

**Proposition 3.6.** If $\tau, \sigma$ are r.o.-equivalent topologies on a space $X$, the identity mapping $i: (X, \tau) \to (X, \sigma)$ is $s$-perfect.

**Proof.** Point inverses are singletons and are therefore $S$-sets. We show that $i$ is an $s$-closed mapping.

Let $A$ be a subset of $X$ and suppose $i(x) \notin (\text{cl}_\tau A)$. Then $x \notin \text{cl}_\tau A$ and there exists $S \in \mathcal{S}_\tau(X, \tau)$ such that $S^\tau \cap A = \emptyset$. By 1.1 of [5], $S^\tau = \overline{S}^\sigma \in \mathcal{S}_\sigma(x)$ and $\emptyset = S^\tau \cap A = \overline{S}^\sigma \cap A$. Thus $i(x) \notin (\text{cl}_\sigma (i(A)))$ and the proof is complete.

**Corollary 3.7.** If $\tau, \sigma$ are r.o.-equivalent topologies on a space $X$, then a subset $A$ of $X$ is an $S$-set relative to $(X, \tau)$ if and only if $A$ is an $S$-set relative to $(X, \sigma)$.

**Proof.** Let $i: (X, \tau) \to (X, \sigma)$ be the identity mapping so that $i^{-1}: (X, \sigma) \to (X, \tau)$ is also the identity mapping. By Proposition 3.6, $i$ and $i^{-1}$ are $s$-perfect mappings and by Proposition 3.3, both preserve $S$-sets.
Remark 3.8. Let $g: X \to X_s$ denote the semiregularization of a space $X$. Then $X$ and $X_s$ are r.o.-equivalent, $g$ and $g^{-1}$ are s-perfect, and $A$ is an S-set in $X$ if and only if $A$ is an S-set in $X_s$.

N. Levine has shown that $S(X, \tau) = S(X, \sigma)$ implies $\tau = \sigma$ [4]. We conclude that a bijective semi-open and irresolute mapping is a homeomorphism. Compare [1].

Proposition 3.9. Suppose $f: X \to Y$ maps an ordinary point to an S-point. If either (i) $f$ is continuous or (ii) $Y$ is S-closed, then $f$ is not s-perfect.

Proof. Let $x$ be an ordinary point of $X$ whose image, $y = f(x)$, is an S-point. There exists $S \in S_x$ such that $x \notin (S)$. $\mathcal{F} = \{N \cap S^0|N \in \pi_x\}$ is a filter base on $X$. sad $\mathcal{F} \subseteq \text{sad } \pi_x \subseteq \{x\}$; but $S_x = X \setminus (S)^0 \in S_x$ and for $N \in \pi_x$, $\emptyset = S_x \cap (N \cap (S)^0).$ Thus sad $\mathcal{F} = \emptyset$.

If $Y$ is S-closed, $f(\mathcal{F})$ s-accumulates at some $z \in Y$; and, if $f$ is continuous, $f(\mathcal{F})$ s-converges to $y$ (y is an S-point). In either case, sad $f(\mathcal{F}) \neq \emptyset = f(\text{sad } \mathcal{F})$. By Lemma 3.2, $f$ is not s-perfect.

Corollary 3.10. If $f: X \to Y$ is an s-perfect mapping and either (i) $f$ is continuous and $Y$ is e.d. or (ii) $Y$ is S-closed, then $X$ is e.d.

Proof. $Y$ is e.d., so that Remark 2.6 and Proposition 3.9 imply that every $x \in X$ is an S-point. Therefore $X$ is e.d. by Remark 2.6.

Corollary 3.11. If $f: X \to Y$ is a continuous mapping of an $H$-closed space into an S-closed space, then $f$ is s-perfect if and only if $X$ is e.d.

Proof. Sufficiency is given by Corollary 3.5 and necessity follows from Corollary 3.10.

4. Related results. In [7], Thompson showed that every compact e.d. space is S-closed. We have shown that every $H$-closed e.d. space is S-closed and it is well known that there exist noncompact, $H$-closed e.d. spaces [5]. Thus the class of S-closed spaces properly contains the class of compact e.d. spaces.

G. Viglino defined a space to be C-compact if every closed subset of $X$ is an $H$-set. In so doing, Viglino introduced a class of spaces which properly contains the class of compact spaces and is properly contained in the class of minimal Hausdorff (and hence $H$-closed) spaces [10]. It therefore seems natural to ask whether or not a new class of noncompact spaces, which is properly contained in the class of S-closed spaces, can be found in similar fashion. Our next proposition provides a negative answer to this question.

Proposition 4.1. Every closed subset of a space $X$ is an S-set if and only if $X$ is compact and e.d.

Proof of sufficiency. If $C$ is a closed subset of a compact e.d. space and $\{S_a\}_{a \in A}$ is an s.o. cover of $C$, then $\{S_a\}_{a \in A}$ is an open cover of a compact set so that $C$ is contained in some finite union of elements.

Proof of necessity. If every closed subset of $X$ is an S-set, then, in particular, $X$ is S-closed so that $X$ is e.d. Thus $X$ is a Urysohn space. Moreover, every closed
subset of $X$ is an $H$-set so that $X$ is normal [6, Theorem 3.3]. Thus $X$ is compact by [6, (2.17)] and the proof is complete.

**Corollary 4.2.** The continuous irresolute image of a compact e.d. space is compact and e.d.

**Proof.** The compactness of a continuous image of a compact space is well known. To obtain the e.d. property, apply Propositions 3.1 and 4.1.

**Remark 4.3.** A. Gleason proved that every compact space is the continuous perfect irreducible image of a compact e.d. space [3]. Proposition 3.4 and Corollary 4.2 show that the associated mappings are necessarily $s$-perfect but need not be irresolute.

**Bibliography**


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