ON NOWHERE DENSE CCC P-SETS

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Abstract. We prove that no compact Hausdorff space can be covered by nowhere dense ccc P-sets. As an application it follows that if X is a compact Hausdorff space with a nonisolated P-point then X × K is not homogeneous for any compact ccc space K.

1. Introduction. All spaces under discussion are Tychonoff.

A subset B of a space X is called a P-set whenever the intersection of countably many neighborhoods of B is again a neighborhood of B. It is known that no compact space of π-weight ω₁ can be covered by nowhere dense P-sets [KvMM]. In addition, there is a compact space of weight ω₂ which can be covered by nowhere dense P-sets [KvMM]. In this note we will show that no compact space can be covered by nowhere dense ccc P-sets. As a consequence it follows that if X is a compact space with a nonisolated P-point then X × K is not homogeneous for any compact ccc space K.

2. Independent matrices. Let X be a space. An indexed family \( \{ A^i_j : i \in I, j \in J \} \) is called an I by J independent matrix for X provided that

(a) each \( A^i_j \) is an open \( F_{α} ; \)
(b) if \( i \in I \) and \( j_0, j_1 \) are distinct elements of J then \( A^i_{j_0} \cap A^i_{j_1} = \emptyset ; \)
(c) if \( F \subset I \) is finite and \( ϕ : F \to J \) then \( \bigcap_{i \in F} A^i_{ϕ(i)} \neq \emptyset . \)

This concept, in a slightly different form, is due to Kunen.

In [vM₃] it was shown that each compact space in which each nonempty \( G_{β} \) has nonempty interior contains an \( ω₁ \) by \( ω₁ \) independent matrix. We need a generalization of this result. As usual, a space is called ccc if each pairwise disjoint collection of nonempty open sets is countable. A space is nowhere ccc if no point has a ccc neighborhood.

2.1. Theorem. Suppose that X is nowhere ccc. Then X contains an \( ω₁ \) by \( ω₁ \) independent matrix.

Proof. For each finite subset \( F \subset ω₁ \) (possibly empty) we will define an open \( F_{α} , \)
\( C_{F} \subset X , \) such that

(i) \( C_{F \cup \{ α \}} \subset C_{F} \) for all \( \max F < α < ω₁ ; \)
(ii) \( C_{F \cup \{ α \}} \cap C_{F \cup \{ β \}} = \emptyset \) if \( \max F < α < β < ω₁ \)
(as usual, an ordinal is the set of smaller ordinals; we define \( \max \emptyset = -1 \).

We will induct on the cardinality of F. Define \( C_{\emptyset} = X . \)

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Suppose that we have defined \( C_F \) for all \( F \subseteq \omega_1 \) of cardinality \( n \). Let \( \{ C_{F \cup \{a\}} : \max F < a < \omega_1 \} \) be a “faithfully indexed” collection of pairwise disjoint nonempty open \( F_a \)'s of \( C_F \). This completes the induction.

**Fact.** \( C_F \cap C_G \neq \emptyset \rightarrow (F \subseteq G) \lor (G \subseteq F) \).

We induct on the cardinality of \( |F| + |G| \). If \( |F| + |G| = 1 \) then there is nothing to prove. Suppose that we have proved the Fact for all finite sets \( F, G \subseteq \omega_1 \) satisfying \( |F| + |G| < n - 1 \). Now take finite sets \( S, T \subseteq \omega_1 \) so that \( |S| + |T| < n \). Define \( S' = S - \{ \max S \} \). By (i) we have that \( C_S \subseteq C_{S'} \) and consequently \( C_{S'} \cap C_T \neq \emptyset \). By induction hypothesis, \( S' \subseteq T \lor T \subseteq S' \). If \( T \subseteq S' \) then we are done, so we may assume that \( S' \subseteq T \). Define \( T' = T - \{ \max T \} \). By precisely the same argumentation we may conclude that \( T' \subseteq S \). Then clearly

\[
(S \cap T) \cup \{ \max S \} = S \quad \text{and} \quad (S \cap T) \cup \{ \max T \} = T.
\]

If \( \max S \in T \) or \( \max T \in S \) then there is nothing to prove. Assume that \( \max S < n \).

Now, let us assume that \( U^a_{\beta} \cap U^\gamma_{\beta} \neq \emptyset \) for some \( \beta \neq \gamma \). Without loss of generality assume that \( f^{-1}(\langle \alpha, \beta \rangle) \subset f^{-1}(\langle \alpha, \gamma \rangle) \). There are finite sets \( F_0, F_1 \subseteq \omega_1 \) so that

(a) \( C_{F_0 \cup \{ f^{-1}(\langle \alpha, \beta \rangle) \}} \cap C_{F_1 \cup \{ f^{-1}(\langle \alpha, \gamma \rangle) \}} \neq \emptyset \);

(b) \( \max F_0 < f^{-1}(\langle \alpha, \beta \rangle) \) and \( f[F_0] \cap \{ (\alpha) \times \omega_1 \} = \emptyset \);

(c) \( \max F_1 < f^{-1}(\langle \alpha, \gamma \rangle) \) and \( f[F_1] \cap \{ (\alpha) \times \omega_1 \} = \emptyset \).

Since \( f^{-1}(\langle \alpha, \gamma \rangle) \notin F_0 \cup \{ f^{-1}(\langle \alpha, \beta \rangle) \} \), by the Fact, \( F_0 \cup \{ f^{-1}(\langle \alpha, \beta \rangle) \} \subset F_1 \cup \{ f^{-1}(\langle \alpha, \gamma \rangle) \} \). Therefore \( f^{-1}(\langle \alpha, \beta \rangle) \in F_1 \), since \( f^{-1}(\langle \alpha, \beta \rangle) \neq f^{-1}(\langle \alpha, \gamma \rangle) \). However, this contradicts (c).

Take \( \alpha_1, \ldots, \alpha_n < \omega_1 \) so that \( \alpha_i \neq \alpha_j \) for \( i \neq j \). In addition, take \( \beta_i < \omega_1 \) (\( i < n \)) arbitrarily. Put \( \gamma_i = f^{-1}(\langle \alpha_i, \beta_i \rangle) \) and without loss of generality assume that \( \gamma_1 < \gamma_2 < \cdots < \gamma_n \). Then \( C_{\langle \gamma_1, \ldots, \gamma_n \rangle} \subseteq U^\gamma_{\beta_1} \cap \cdots \cap U^\gamma_{\beta_n} \) and since \( C_{\langle \gamma_1, \ldots, \gamma_n \rangle} \neq \emptyset \) we find that \( U^\gamma_{\beta_1} \cap \cdots \cap U^\gamma_{\beta_n} \neq \emptyset \). \( \square \)

**3. The first application.** A point \( x \in X \) is called a weak P-point provided that \( x \notin \overline{F} \) for each countable \( F \subseteq X - \{ x \} \). Kunen [K] proved that there is a weak P-point in \( \omega^* \) (= \( \beta\omega \setminus \omega \)). Subsequently van Mill [vM1] proved that there is a weak P-point in each compact F-space of weight \( 2^n \) in which each nonempty \( G_\beta \) has nonempty interior (an F-space is a space in which each cozero set is C*-embedded). Bell [B] has since shown that the weight condition is superfluous. Using Theorem 2.1 by precisely the same technique as in [vM1] we obtain the following generalization.

**3.1. Theorem.** Each compact nowhere ccc F-space contains a weak P-point.

**4. The main result.** In this section we derive our main result. The techniques of proof used in the following lemma is the same as in [vM1], [vM2].
4.1. Lemma. No compact nowhere ccc space can be covered by ccc P-sets.

Proof. Let $X$ be a compact nowhere ccc space. Clearly $X$ is not finite, so there is a collection $\{V_n: n < \omega\}$ of (faithfully indexed) pairwise disjoint nonempty open $F_\sigma$ subsets of $X$. For each $n < \omega$ let $\{U_{\alpha}^i(n): \alpha < \omega_1, i < \omega\}$ be an $\omega_1$ by $\omega$ independent matrix for $V_n$ (Theorem 1.1). Notice that each $U_{\alpha}^i(n)$ is an open $F_\sigma$ of $X$. Put $F = \{A \subset X: \forall n < \omega \forall i < n \exists \alpha < \omega_1 \text{ such that } U_{\alpha}^i(n) \subset A\}$. It is clear that $F$ has the finite intersection property, so there is an $x \in \bigcap_{F \in F} F$. We claim that $x \notin K$ for each ccc P-set $K$. Indeed, let $K \subset X$ be any ccc P-set. Since $K$ is ccc for each $n < \omega$ and for each $i < n$ there is an $\alpha(n, i) < \omega_1$ so that $U_{\alpha(n, i)}^i(n) \cap K = \emptyset$. Put $F = \bigcup_{n < \omega} \bigcup_{i < n} U_{\alpha(n, i)}^i(n)$. Then $F \in F$ and $F$ is an open $F_\sigma$ being the union of countably many open $F_\sigma$'s. Also, $F \cap K = \emptyset$. Since $K$ is a P-set, it also follows that $F \cap K = \emptyset$. We conclude that $x \notin K$. \[\square\]

We now come to our main result.

4.2. Theorem. No compact space can be covered by ccc nowhere dense P-sets.

Proof. Let $X$ be a compact space and suppose that $X$ can be covered by ccc nowhere dense P-sets. Let $U \subset X$ be nonempty and open and suppose that $U$ is ccc. Let $B$ be a nowhere dense P-set meeting $U$. Since $B \cap U$ is nowhere dense in $U$ the fact that $U$ is ccc implies that there is a countable family $\mathcal{G}$ of compact subsets of $U - B$ so that $\bigcup \mathcal{G}$ is dense in $U$. However, this is impossible since $B$ is a P-set. So $U$ is not ccc. But now the assumption that $X$ can be covered by ccc nowhere dense P-sets contradicts Lemma 4.1. \[\square\]

5. Another application. A space $X$ is called homogeneous provided that for all $x, y \in X$ there is an autohomeomorphism $\varphi$ from $X$ onto $X$ mapping $x$ onto $y$. It is well known that although $X$ is not homogeneous the product $X \times K$ can be homogeneous for certain $K$ (for example, let $X$ be a convergent sequence and let $K$ be the Cantor set). This makes the following straightforward corollary to Theorem 4.2 of some interest.

5.1. Corollary. Let $X$ be a compact space having a nonisolated P-point. Then $X \times K$ is not homogeneous for any compact ccc nonempty space $K$.

Proof. Let $x$ be a nonisolated P-point of $X$. Then $\{x\} \times K$ is a ccc nowhere dense P-set of $X \times K$. Take any $\langle x, y \rangle \in \{x\} \times K$. By Theorem 4.2 there is a point $\langle p, q \rangle \in X \times K$ so that $\langle p, q \rangle \notin E$ for any nowhere dense ccc P-set $E \subset X \times K$. It is clear that no autohomeomorphism of $X \times K$ can map $\langle x, y \rangle$ onto $\langle p, q \rangle$. \[\square\]

6. Questions. Since there is a compact space $X$ of weight $\omega_2$ which can be covered by nowhere dense P-sets (which all have to have cellularity at most $\omega_2$), Theorem 4.2 suggests the following question:

6.1. Question. Is there a compact space $X$ which can be covered by nowhere dense P-sets of cellularity at most $\omega_1$?
Since Frankiewicz and Mills [FM] have shown that Con(\(\text{ZFC} + \omega^*\) can be covered by nowhere dense \(P\)-sets) the question naturally arises whether it is consistent that \(\omega^*\) can be covered by nowhere dense \(P\)-sets of cellularity at most \(\omega_1\).

Let us answer this question.

6.2. **Proposition.** \(\omega^*\) cannot be covered by nowhere dense \(P\)-sets of cellularity at most \(\omega_1\).

**Proof.** Under CH the result follows from [KvMM]. So assume \(\neg\text{CH}\). Kunen [K] proved that (in ZFC) there is a \(2^\omega\) by \(2^\omega\) independent matrix of clopen subsets of \(\omega^*\). Since \(\omega_1 < 2^\omega\) we can use the same proof as in Lemma 4.1 to get a point \(x \in \omega^*\) so that \(x \notin B\) for any \(P\)-set \(B\) of cellularity at most \(\omega_1\). \(\Box\)

Let us finally notice that Proposition 5.1 suggests the following question.

6.3. **Question.** Let \(X\) be a compact space having a nonisolated \(P\)-point and let \(K\) be compact. Is \(X \times K\) not homogeneous?

**References**


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