WHEN U(κ) CAN BE MAPPED ONTO U(ω)

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Abstract. U(κ) can be mapped onto U(ω) iff cf(κ) = ω or κ > 2ω.

0. Introduction. In this note we show that U(κ) can be mapped onto U(ω) if and only if cf(κ) = ω or κ > 2ω. As a consequence it follows that CH is equivalent to the statement that U(ω₁) can be mapped onto U(ω). That U(ω) is not always a continuous image of U(ω₁) is known, [B], however, as far as I know, it was unknown that U(ω) is not a continuous image of U(ω₁) under ¬CH.

1. Conventions. Cardinals carry the discrete topology. If κ is a cardinal then βκ denotes the Čech-Stone compactification of κ. The subspace

\{ p ∈ βκ: if F ⊆ p then |F| = κ \}

of βκ is denoted by U(κ). It is easy to see that U(κ) is compact. For more information on βκ and U(κ) see [CN].

2. The construction.

2.1. Lemma. If cf(κ) = ω then U(κ) can be mapped onto U(ω).

Proof. Let κ = ∑ₙ<ω κₙ where, for each n, κₙ < κ. Define f: κ → ω by f(α) = n iff α ∈ κₙ and let βf: βκ → βω be the Stone extension of f. It is routine to verify that βf(U(κ)) = U(ω). □

2.2. Remark. This lemma is known of course, see for example [vD].

2.3. Lemma. If κ > 2ω then U(κ) can be mapped onto U(ω).

Proof. Let (Aₐ: a < 2ω) be a (faithfully indexed) partition of κ into 2ω subsets of cardinality κ. Define f: κ → 2ω by f(α) = μ iff α ∈ Aμ and let βf: βκ → β(2ω) be the Stone extension of f. It is routine to verify that βf(U(κ)) = U(ω). Since U(ω) has clearly weight 2ω and since β(2ω) maps onto each compact space of weight at most 2ω, we conclude that U(κ) can be mapped onto U(ω). □

2.4. Lemma. If ω < cf(κ) < κ < 2ω then U(ω) is not a continuous image of U(κ).

Proof. Suppose, to the contrary, that f maps U(κ) onto U(ω). Since there is clearly a compactification of ω with I = [0, 1] as remainder, there is a map g from U(ω) onto I. Let h: U(κ) → I be the composition of f and g. In addition, let h: βκ → I extend h.
Take \( s \in I \) arbitrarily. Then \( g^{-1}(\{s\}) \) is a nonempty \( G_\delta \) in \( U(\omega) \) and consequently has nonempty interior, [CN, 14.17]. Therefore, \( f^{-1}g^{-1}(\{s\}) \) has nonempty interior (in \( U(\kappa) \)) and consequently we can find a subset \( E \subseteq \kappa \) so that
\[
\emptyset \neq E \cap U(\kappa) \subseteq f^{-1}g^{-1}(\{s\}).
\]

**Claim.** If \( n < \omega \) then \(|\{\alpha \in E: \tilde{h}(\alpha) \notin (s - 1/n, s + 1/n)\}| < \kappa \). Suppose, to the contrary, that \( F = \{\alpha \in E: \tilde{h}(\alpha) \notin (s - 1/n, s + 1/n)\} \) has cardinality \( \kappa \).

Take a point \( x \in F \cap U(\kappa) \). By continuity of \( \tilde{h} \), the point \( \tilde{h}(x) \notin (s - 1/n, s + 1/n) \). This implies that \( x \notin (E \cap U(\kappa)) - f^{-1}g^{-1}(\{s\}) \), which is impossible.

Since \( \text{cf}(\kappa) > \omega \) the claim implies that we can find \( \kappa_x \in E \) so that \( \tilde{h}(\kappa_x) = s \).

This is a contradiction since \( \kappa < 2^\omega = |I| \). □

2.5. **Corollary.** CH is equivalent to the statement that \( U(\omega_1) \) can be mapped onto \( U(\omega) \).

**Proof.** Since \( \omega_1 \) has uncountable cofinality this immediately follows from Lemmas 2.3 and 2.4. □

REFERENCES


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