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K-THEORY OF AZUMAYA ALGEBRAS

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ABSTRACT. Quillen has defined a K-theory for symmetric monoidal categories. We show that Quillen's groups agree with the groups K_0, K_1, and K_2 defined by Bass. Finally, we compute the K-theory of the Azumaya algebras over a commutative ring.

The purpose of this paper is to advertise the K-theory of symmetric monoidal categories, and to compute the K-theory of the category of Azumaya R-algebras. The point is that Quillen's theory (introduced in [6]) is a natural generalization of the "classical" theory for K_0, K_1, K_2 defined by Bass in [2], [3], [4]. On the other hand, it provides a wealth of examples of infinite loop spaces (see [1], [9], [10], [12] and [13]).

A symmetric monoidal category is a category S with a unit $0: * \to S$ and a product $\square$: $S \times S \to S$ which is commutative and associative up to coherent natural isomorphism; the precise definition may be found in [7]. We shall be especially interested in the following examples (from [2]):

(1) **P**, the fin. gen. projective modules over a ring R. The product $\square$ is direct sum, and we consider only isomorphisms.

(2) **FP**, the fin. gen. faithful projective modules over a commutative ring R. The product $\square$ is the tensor product, and the arrows are isomorphisms.

(3) **Pic**, the full subcategory of **FP** of rank one projective modules.

(4) **Az**, the Azumaya algebras over a commutative ring R. The arrows are R-algebra isomorphisms, and the product is the tensor product. If R is a field an Azumaya algebra is just a central simple algebra.

In the language of [3, Chapter VII], a symmetric monoidal category is a "category with product $\square$", with the additional condition that there be a special object 0 and natural isomorphisms $0 \square s \cong s \cong s \square 0$ satisfying the coherence conditions on page 159 of [7]. Groups $K_i^{\det}(S)$ ($i = 0, 1, 2$) were defined and studied in [2], [3] and [4], using only the objects, isomorphisms and product of the category S.

We will restrict our attention to the category *SMCat* of small symmetric monoidal categories and relaxed morphisms. We require in addition that every symmetric monoidal category S in *SMCat* satisfies (i) every arrow is an isomorphism, and (ii) every translation $s\square$: $\mathrm{Aut}(t) \to \mathrm{Aut}(s \square t)$ is an injection. The categories **P**, **FP**, **Pic**, **Az** all belong to *SMCat*, as do the categories:

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(5) $\mathbf{Quad}^\lambda(A, \Lambda)$, the category of nonsingular (λ, Λ)-quadratic A-modules defined in [4]. Here A is a ring with involution, λ is a central element of A satisfying $\lambda\bar{\lambda} = 1$, and Λ is a additive subgroup of $\{a \in A: a = -\lambda\bar{a}\}$ containing $\{a - \lambda\bar{a}\}$ and closed under $r \mapsto ar\bar{a}$. The product is direct sum. The principal goal of [4] was to calculate the groups $K_1^{\det}(\mathbf{Quad}^\lambda)$ for various (A, λ, Λ).

(6) **Ens**, the category of finite sets and their isomorphisms, the product being disjoint union. It is easy to see that $K_0^{\det}(\mathbf{Ens}) = \mathbf{Z}$, $K_1^{\det}(\mathbf{Ens}) = \{\pm 1\}$; it is known (see [1]) that the Quillen K-groups $K_i(\mathbf{Ens})$ are the "stable stems" $\pi_i^s = \underrightarrow{\operatorname{colim}}\, \pi_{n+i}(S^n)$. The "free module" functor from **Ens** to $\mathbf{P(Z)}$ induces the map $\pi_*^s \to K_*(\mathbf{Z})$.

1. Quillen K-theory. In [6], Quillen defined groups $K_*(S)$ for every S in *SMCat*. This is achieved by associating to every S in *SMCat* a new symmetric monoidal category $S^{-1}S$ (not in *SMCat*) properly containing S. Applying geometric realization yields a topological space $BS^{-1}S$; the groups $K_*(S)$ are defined to be the homotopy groups $\pi_*(BS^{-1}S)$. It is shown in [6] that the groups $K_*(\mathbf{P})$ coincide with the algebraic K-groups $K_*(R)$ of the underlying ring R.

One pleasing property of these topologically defined groups is that they agree with the classically defined K-groups. Classically, $K_0^{\det}(S)$ is the group completion of the abelian monoid of isomorphism classes of objects of S. Bass (in [2], [3]) defined $K_1^{\det}(S)$ to be the direct colimit of the groups $H_1(\operatorname{Aut}_S(s)) = \operatorname{Aut}(s)/[\operatorname{Aut}(s), \operatorname{Aut}(s)]$.

Proposition 1. *Quillen's groups $K_i(S)$ agree with Bass's groups $K_i^{\det}(S)$ for $i = 0, 1$.*

Proof. From [6] we know that $H_*(BS^{-1}S) = \operatorname{colim} H_*(BS)$, where the colimit is taken over the directed set of (isomorphism classes of) objects s in S under translation. For $* = 0$ we obtain the K_0 result. Reading this for $* = 1$ yields $K_1(S) = \pi_1(BS^{-1}S) = H_1(B_0S^{-1}S) = \operatorname{colim} H_1(B\operatorname{Aut}(s)) = K_1^{\det}(S)$.

Remark. In [4], Bass defined groups $K_2^{\det}(S)$. In the next section we will show that this agrees with the $K_2(S)$ of Quillen.

Another pleasing property is that the spaces $BS^{-1}S$ are infinite loop spaces. This follows from the fact that $\pi_0 BS^{-1}S$ is the group $K_0^{\det}(S)$ and Proposition 2 below. For example, $B\,\mathbf{Ens}^{-1}\mathbf{Ens}$ is the space $\Omega^\infty\Sigma^\infty$, and $B\mathbf{P}^{-1}\mathbf{P}$ is the space $K_0(R) \times B\,\mathrm{Gl}(R)^+$ (see page 91 of [1]).

Proposition 2. *If T is a small monoidal category, BT is a homotopy associative H-space. If T is symmetric monoidal, BT is also homotopy commutative, and BT is an infinite loop space if and only if $\pi_0(BT)$ is an abelian group.*

Remark. There is a simple, purely algebraic definition of $\pi_0(BT)$. If T is a small symmetric monoidal category, define $\pi_0 T$ to be the set of objects of T, modulo the equivalence relation generated by requiring $s \sim t$ whenever there is an arrow from s to t. The product $\square$ makes $\pi_0 T$ an abelian monoid. If T is in *SMCat*, $\pi_0 T$ is the monoid of isomorphism classes of objects. Since $\pi_0 T$ is $\pi_0(BT)$, the topological space BT is an infinite loop space iff $\pi_0 T$ is a group.

PROOF. The functor $\square$: $T \times T \to T$ induces $B\square$: $BT \times BT \cong B(T \times T) \to BT$, making BT an H-space. Associativity (and commutativity in the symmetric case) of $\square$ up to natural equivalence translates directly into homotopy associativity (and commutativity) of BT. To determine when BT is an infinite loop space, we use Segal's machine [12]; this is appropriate since Thomason has shown in [13] that BT is the initial space of a Γ-space. By [15, p. 461], $B\square$ has a homotopy inverse iff $\pi_0(BT)$ is a group, and by [12] this is necessary and sufficient for BT to be an infinite loop space.

REMARK. We could have also used May's machine. In the relevant vocabulary, BT is an A_∞ space if T is monoidal, and BT is an E_∞ space if T is symmetric monoidal. This was shown in [9]. The above formulation of Proposition 2 was shown to me by Z. Fiedorowicz.

The usefulness of Proposition 2 is that some of the S in *SMCat* already have a group for $\pi_0 S$. In this case, the natural map $BS \to BS^{-1}S$ is a homotopy equivalence (it is an infinite loop space map which is a homology isomorphism). For example, this is true of S = **Pic**. It follows from [2] or [3] that B **Pic** $\simeq \text{Pic}(R) \times BU(R)$, where $\text{Pic}(R)$ is the Picard group of the commutative ring R, and $U(R)$ is the group of units of R. We have the

COROLLARY. K_0 **Pic** $= \text{Pic}(R)$, K_1 **Pic** $= U(R)$, *and the groups* K_***Pic** *are zero for* $* \geq 2$.

2. The plus construction and K_2. If the category S has a countable, cofinal subcategory, we can construct a group $\text{Aut}(S)$ playing the role that $\text{Gl}(R)$ does for **P**. The construction is given on page 355 of [3], although the constructions of [2, p. 25], [4, p. 197], and [14] may be used where appropriate.

The groups $\text{Aut}(S)$ are easy to compute in the sample categories given in the introduction. The free modules in **P** and **FP** allow us to take $\text{Aut}(\mathbf{P}) = \text{Gl}(R)$, $\text{Aut}(\mathbf{FP}) = \text{Gl}_\otimes(R) = \text{colim}\{\text{Gl}_n(R);\ \alpha \mapsto \alpha \otimes I\}$. Aut(**Pic**) is just $U(R)$. The matrix rings in **Az** allow us to take Aut(**Az**) to be the direct colimit of the R-algebra automorphisms of the $M_n(R)$. We have $\text{Aut}(\mathbf{Quad}^\lambda(A, \Lambda)) = U^\lambda(A, \Lambda) = \text{colim}\ U^\lambda_{2n}(A, \Lambda)$ and $\text{Aut}(\mathbf{Ens}) = \Sigma_\infty = \text{colim}\ \Sigma_n$.

PROPOSITION 3. *Suppose that* S *has a countable, cofinal subcategory, so that* $\text{Aut}(S)$ *exists. Then the commutator subgroup* E *of* $\text{Aut}(S)$ *is a perfect, normal subgroup, so the plus construction may be applied to* $B\ \text{Aut}(S)$. *The resulting space is the basepoint component of* $BS^{-1}S$, *i.e.,* $BS^{-1}S \simeq K_0(S) \times B\ \text{Aut}(S)^+$. *Moreover,* $K_1(S) = \text{Aut}(S)/E$.

PROOF. As E is a direct colimit, every element of E is a product of elements, each represented by a commutator $[\alpha, \beta]$ in some $\text{Aut}(s)$. We compute in $\text{Aut}(s \square s \square s)$ that $[\alpha, \beta] \square 1 \square 1 = [\alpha \square \alpha^{-1} \square 1,\ \beta \square 1 \square \beta^{-1}]$, which represents an element of $[E, E]$ by the Abstract Whitehead Lemma on page 351 of [3]. This shows that E is perfect, so that f: $B\ \text{Aut}(S) \to B\ \text{Aut}(S)^+$ exists and is any acyclic map with $\ker(\pi_1 f) = E$. If we copy the telescope construction of [6], we obtain such an acyclic map from $B\ \text{Aut}(S)$ to the basepoint component of $BS^{-1}S$, proving the proposition.

We are now in a position to compare Quillen's K_2 to Bass's $K_2^{\det}$. In Appendix A to [4], Bass defined $K_2^{\det}(S)$ to be the direct colimit of the groups $H_0(\mathrm{Aut}(s); [\mathrm{Aut}(s), \mathrm{Aut}(s)])$.

We remark that when $\mathrm{Aut}(S)$ exists we have $K_2^{\det}(S) = H_2(E)$. This may be seen by reading the proof of (A.6) on page 200 of [4]. In this case, $K_2^{\det}(S)$ may also be interpreted as the kernel of a universal central extension of the perfect group E (as in [11]).

THEOREM 4. *Quillen's $K_2(S)$ is the same as Bass's $K_2^{\det}(S)$.*

PROOF. Any S in *SMCat* is the direct colimit of full subcategories which are countable, and hence for which $\mathrm{Aut}(S)$ exists. As Bass's and Quillen's groups both commute with direct colimits, we are reduced to proving the theorem when $\mathrm{Aut}(S)$ exists. In this case we have to show that $K_2(S) = H_2(E)$. We will use a modification of the proof of Proposition 4.12 in [9], which is essentially due to D. W. Anderson.

There is a homotopy fibration $BE \to B\,\mathrm{Aut}(S) \to B(K_1S)$. Since K_1S is an abelian group, $B(K_1S)$ is an Eilenberg-Mac Lane space. The map $B\,\mathrm{Aut}(S) \to B(K_1S)$ factors through an H-space map $B\,\mathrm{Aut}(S)^+ \to B(K_1S)$ by universality of the plus construction. If F denotes the fiber of the latter map, there is a map of fibrations:

$$\begin{array}{ccccc} BE & \to & B\,\mathrm{Aut}(S) & \to & B(K_1S) \\ \downarrow & & \downarrow & & \| \\ F & \to & B\,\mathrm{Aut}(S)^+ & \to & B(K_1S). \end{array}$$

The action of $K_1S = \pi_1 B(K_1S)$ on BE is trivial for the same reasons given in [9]: if $y \in \mathrm{Aut}(s)$ represents $[y] \in K_1S$ and $z \in H_*(BE)$, we can choose a subgroup $\mathrm{Aut}(t)$ of $\mathrm{Aut}(s \,\square\, t)$ for some t so that z is in the image of $H_*(B[\mathrm{Aut}(t), \mathrm{Aut}(t)])$. As y commutes with $\mathrm{Aut}(t)$, $[y]$ acts trivially on z. On the other hand, the action of K_1S on $H_*(F)$ is trivial because F is connected and is the fiber of an H-space map (see [5, p. 16-09]). It follows by the Comparison Theorem (in [8]) that $H_*(E) = H_*(BE) \to H_*(F)$ is a homology isomorphism. On the other hand, F is simply connected, so $H_2(F) \cong \pi_2(F) \cong \pi_2(B\,\mathrm{Aut}(S)^+) = K_2(S)$.

We will need the following result which is implicit in [10, p. 96], and was pointed out in [14]. The proof involves a comparison of the groups $\mathrm{Gl}(R)$ and $\mathrm{Gl}_{\otimes}(R)$.

PROPOSITION 5. $K_*(\mathbf{FP}) = \mathbf{Q} \otimes K_*(\mathbf{P}) = \mathbf{Q} \otimes K_*(R)$ *for* $* > 1$, *while* $K_0(\mathbf{FP}) = U^+(\mathbf{Q} \otimes K_0(R))$ *in the notation of* [3, *p*. 516].

3. Azumaya algebras. In this section we compute the groups $K_*\mathbf{Az}$. The computation was inspired by the calculations of [2] and [14]. I am indebted to C. McGibbon and J. Neisendorfer for suggesting the use of the Comparison Theorem in the proof.

There is a functor End: $\mathbf{FP} \to \mathbf{Az}$ in *SMCat*, which sends a faithful projective R-module P to its endomorphism ring $\mathrm{End}(P)$, and sends the automorphism α of P to conjugation by α. This induces a map End: $\mathrm{Gl}_{\otimes}(R) \to \mathrm{Aut}(\mathbf{Az})$. The following result is proven in [2] and on page 74 of [3]:

PROPOSITION 6 (ROSENBERG-ZELINSKY). *There is an exact sequence*

$$1 \to U(R) \to \mathrm{Gl}_{\otimes}(R) \xrightarrow{\mathrm{End}} \mathrm{Aut}(\mathbf{Az}) \to T\,\mathrm{Pic}(R) \to 1,$$

where $T\,\mathrm{Pic}(R)$ *is the torsion subgroup of* $\mathrm{Pic}(R)$.

We consider $T\,\mathrm{Pic}(R)$ to represent outer automorphisms, and would like a category of inner automorphisms. We define **In** to be the image of End. **In** is the monoidal subcategory of **Az** whose objects are the End(P), and whose arrows are "inner" automorphisms. The group Aut(**In**) = colim In Aut($M_n(R)$) is the group $\mathrm{PGl}_{\otimes}(R) = \mathrm{Gl}_{\otimes}(R)/U(R)$ (cf. pages 108, 119 of [**2**] and page 74 of [**3**]). We thus have short exact sequences of groups $1 \to U(R) \to \mathrm{Gl}_{\otimes}(R) \to \mathrm{PGl}_{\otimes}(R) \to 1$ and $1 \to \mathrm{PGl}_{\otimes}(R) \to \mathrm{Aut}(\mathbf{Az}) \to T\,\mathrm{Pic}(R) \to 1$. The sequence **Pic** → **FP** → **In** gives rise to a commutative diagram of spaces

$$\begin{array}{ccccc} BU(R) & \to & B\,\mathrm{Gl}_{\otimes}(R) & \to & B\,\mathrm{PGl}_{\otimes}(R) \\ \downarrow \simeq & & \downarrow & & \downarrow \\ B_0\,\mathbf{Pic}^{-1}\mathbf{Pic} & \to & B_0\,\mathbf{FP}^{-1}\mathbf{FP} & \xrightarrow{\alpha} & B_0\,\mathbf{In}^{-1}\mathbf{In}. \end{array}$$

The top row is a fibration, and the bottom row is a sequence of infinite loop spaces and infinite loop maps. The left vertical arrow is a homotopy equivalence of infinite loop spaces by Proposition 2. As the bottom composite is trivial, there is an infinite loop space map from $B_0\,\mathbf{Pic}^{-1}\mathbf{Pic}$ to the fiber X of the lower right horizontal map α. Summarizing, there is a map of fibrations

$$\begin{array}{ccccc} BU(R) & \to & B\,\mathrm{Gl}_{\otimes}(R) & \to & B\,\mathrm{PGl}_{\otimes}(R) \\ \downarrow & & \downarrow & & \downarrow \\ X & \to & B_0\,\mathbf{FP}^{-1}\mathbf{FP} & \to & B_0\,\mathbf{In}^{-1}\mathbf{In} \end{array}$$

in which the map $BU(R) \to X$ is an H-map. Now $\mathrm{PGl}_{\otimes}(R)$ acts trivially on $H_*(BU(R))$ because $U(R)$ is central in $\mathrm{Gl}_{\otimes}(R)$ (any element of $\mathrm{PGl}_{\otimes}(R)$ induces the identity map on $BU(R)$). Moreover, $\pi_1(B_0\,\mathbf{In}^{-1}\mathbf{In})$ acts trivially on $H_*(X)$ because α is an H-map and X is connected (see [**5**, p. 16-09]). Hence the Comparison Theorem [**8**, p. 355] applies: as the base and total space maps are homology isomorphisms (by Proposition 3), the infinite loop space map $BU(R) \to X$ is a homology isomorphism, hence a homotopy equivalence. We have proven:

THEOREM 7. $BU(R) \to B\,\mathrm{Gl}_{\otimes}(R)^+ \to B\,\mathrm{PGl}_{\otimes}(R)^+$ *is a homotopy fibration.*

COROLLARY 8. *For* $* \geqslant 3$, $K_*\mathbf{In} \cong K_*\mathbf{FP} \cong \mathbf{Q} \otimes K_*(R)$. *If* $\mu(R)$ *denotes the roots of unity of* R,

$$K_2\mathbf{In} = \mu(R) \oplus K_2\mathbf{FP} = \mu(R) \oplus (\mathbf{Q} \otimes K_2(R)).$$

Finally, $K_1\mathbf{In} = K_1\mathbf{FP}/\mathrm{im}\,U(R)$ *and* $K_0\mathbf{In} = U^+(\mathbf{Q} \otimes K_0(R))/\mathrm{im}(\mathrm{Pic}(R))$.

PROOF. Use the long exact homotopy sequence and the fact that $\pi_*BU = 0$ for $* \neq 1$, as well as Proposition 5. The only subtleties are that in the sequence $0 \to K_2\mathbf{FP} \to K_2(\mathbf{In}) \to U(R) \to K_1\mathbf{FP}$ the left map splits (by divisibility of $K_2\mathbf{FP}$) and that the kernel of the right map is the torsion subgroup $\mu(R)$ of $U(R)$.

THEOREM 9. *There is a long exact sequence in K-theory*:

$$\cdots K_{*+1}\mathbf{Az} \to K_*\mathbf{Pic} \to K_*\mathbf{FP} \to K_*\mathbf{Az} \cdots.$$

In particular: *for* $* \geqslant 3$, $K_*\mathbf{Az} = K_*\mathbf{FP} = \mathbf{Q} \otimes K_*(R)$, $K_2\mathbf{Az} = \mu(R) \oplus K_2\mathbf{FP} = \mu(R) \oplus (\mathbf{Q} \otimes K_2(R))$,

$$K_1\mathbf{Az} = T\operatorname{Pic}(R) \oplus (\mathbf{Q}/\mathbf{Z} \otimes U(R)) \oplus (\mathbf{Q} \otimes SK_1(R)),$$

and $K_0\mathbf{Az} = \operatorname{Br}(R) \oplus U^+(\mathbf{Q} \otimes K_0(R))/\operatorname{im}\operatorname{Pic}(R)$, *where* $\operatorname{Br}(R)$ *is the Brauer group of* R.

PROOF. The map $B\operatorname{Aut}(\mathbf{In}) \to B\operatorname{Aut}(\mathbf{Az})$ is (up to homotopy) a covering space map with fiber the abelian group $T\operatorname{Pic}(R)$. The commutator groups $[\operatorname{Aut}(\mathbf{In}), \operatorname{Aut}(\mathbf{In})]$ and $[\operatorname{Aut}(\mathbf{Az}), \operatorname{Aut}(\mathbf{Az})]$ are isomorphic. Hence we can perform a $T\operatorname{Pic}$-equivariant plus construction on $B\operatorname{Aut}(\mathbf{In})$: for every cell we attach, all translates of the cell are also attached. In this way we obtain the model $B\operatorname{Aut}(\mathbf{In})^+/T\operatorname{Pic}$ for $B\operatorname{Aut}(\mathbf{Az})^+$, and a fibration $T\operatorname{Pic}(R) \to B\operatorname{Aut}(\mathbf{In})^+ \to B\operatorname{Aut}(\mathbf{Az})^+$. This yields $K_*\mathbf{Az}$ for $* > 2$. Bass's analysis of the low-dimensional terms in [2] gives K_0, K_1 and a fibration $B\,\mathbf{Pic}^{-1}\mathbf{Pic} \to B\,\mathbf{FP}^{-1}\mathbf{FP} \to B\,\mathbf{Az}^{-1}\mathbf{Az}$.

REMARK. We have shown that the commutator subgroup E of $\mathrm{PGl}_{\otimes}(R)$ is perfect. In fact, it is the subgroup generated by the images of the elementary matrices in the $\mathrm{Gl}_n(R)$, so the fact that $E = [E, E]$ may be deduced from the fact that elementary matrices are commutators in Gl_n, $n \geqslant 3$. More interesting is the following consequence of Corollary 8: the torsion subgroup of $H_2(E)$ is isomorphic to the roots of unity in the ring R. It would be interesting to find an explicit description of this isomorphism, especially for $R = \mathbf{C}$.

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