\textbf{K-THEORY OF AZUMAYA ALGEBRAS}

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Abstract. Quillen has defined a \(K\)-theory for symmetric monoidal categories. We show that Quillen's groups agree with the groups \(K_0, K_1, K_2\) defined by Bass. Finally, we compute the \(K\)-theory of the Azumaya algebras over a commutative ring.

The purpose of this paper is to advertise the \(K\)-theory of symmetric monoidal categories, and to compute the \(K\)-theory of the category of Azumaya \(R\)-algebras. The point is that Quillen's theory (introduced in [6]) is a natural generalization of the "classical" theory for \(K_0, K_1, K_2\) defined by Bass in [2], [3], [4]. On the other hand, it provides a wealth of examples of infinite loop spaces (see [1], [9], [10], [12] and [13]).

A symmetric monoidal category is a category \(S\) with a unit \(0: \ast \to S\) and a product \(\square: S \times S \to S\) which is commutative and associative up to coherent natural isomorphism; the precise definition may be found in [7]. We shall be especially interested in the following examples (from [2]):

1. \(P\), the fin. gen. projective modules over a ring \(R\). The product \(\square\) is direct sum, and we consider only isomorphisms.
2. \(FP\), the fin. gen. faithful projective modules over a commutative ring \(R\). The product \(\square\) is the tensor product, and the arrows are isomorphisms.
3. \(Pic\), the full subcategory of \(FP\) of rank one projective modules.
4. \(Az\), the Azumaya algebras over a commutative ring \(R\). The arrows are \(R\)-algebra isomorphisms, and the product is the tensor product. If \(R\) is a field an Azumaya algebra is just a central simple algebra.

In the language of [3, Chapter VII], a symmetric monoidal category is a "category with product \(\square\)" with the additional condition that there be a special object \(0\) and natural isomorphisms \(0 \square s \cong s \cong s \square 0\) satisfying the coherence conditions on page 159 of [7]. Groups \(K_i^\text{det}(S)\) \((i = 0, 1, 2)\) were defined and studied in [2], [3] and [4], using only the objects, isomorphisms and product of the category \(S\).

We will restrict our attention to the category \(SMCat\) of small symmetric monoidal categories and relaxed morphisms. We require in addition that every symmetric monoidal category \(S\) in \(SMCat\) satisfies (i) every arrow is an isomorphism, and (ii) every translation \(s\square: \text{Aut}(t) \to \text{Aut}(s \square t)\) is an injection. The categories \(P, FP, Pic, Az\) all belong to \(SMCat\), as do the categories:

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(5) **Quad**^i(A, A), the category of nonsingular \((\lambda, \Lambda)\)-quadratic \(A\)-modules defined in [4]. Here \(A\) is a ring with involution, \(\lambda\) is a central element of \(A\) satisfying \(\lambda^2 = 1\), and \(\Lambda\) is an additive subgroup of \(\{a \in A : a = -\lambda a\}\) containing \(\{a - \lambda a \}\) and closed under \(r \mapsto ar\). The product is direct sum. The principal goal of [4] was to calculate the groups \(K_i^\text{det}(\text{Quad}^i)\) for various \((A, \lambda, \Lambda)\).

(6) **Ens**, the category of finite sets and their isomorphisms, the product being disjoint union. It is easy to see that \(K_0^\text{det}(\text{Ens}) = \mathbb{Z}\), \(K_1^\text{det}(\text{Ens}) = \{\pm 1\}\); it is known (see [1]) that the Quillen \(K\)-groups \(K_i^\text{Ens}\) are the “stable stems” \(\pi_i^* = \text{colim} \pi_{n+i}(S^n)\). The “free module” functor from **Ens** to \(\mathbf{P}(\mathbb{Z})\) induces the map \(\pi_i^* \to K_i^*(\mathbb{Z})\).

1. **Quillen** \(K\)-theory. In [6], Quillen defined groups \(K_i(S)\) for every \(S\) in **SMCat**. This is achieved by associating to every \(S\) in **SMCat** a new symmetric monoidal category \(S^{-1}S\) (not in **SMCat**) properly containing \(S\). Applying geometric realization yields a topological space \(\Sigma S^{-1}S\); the groups \(K_i(S)\) are defined to be the homotopy groups \(\pi_i(\Sigma S^{-1}S)\). It is shown in [6] that the groups \(K_i^\text{det}(\mathbf{P})\) coincide with the algebraic \(K\)-groups \(K_i^*(\mathbb{R})\) of the underlying ring \(\mathbb{R}\).

One pleasing property of these topologically defined groups is that they agree with the classically defined \(K\)-groups. Classically, \(K_0^\text{det}(S)\) is the group completion of the abelian monoid of isomorphism classes of objects of \(S\). Bass (in [2], [3]) defined \(K_i^\text{det}(S)\) to be the direct colimit of the groups \(H_i(\text{Aut}(s)) = \text{Aut}(s)/[\text{Aut}(s), \text{Aut}(s)]\).

**Proposition 1.** Quillen’s groups \(K_i(S)\) agree with Bass’s groups \(K_i^\text{det}(S)\) for \(i = 0, 1\).

**Proof.** From [6] we know that \(H_*(\Sigma S^{-1}S) = \text{colim} H_*(\Sigma S)\), where the colimit is taken over the directed set of (isomorphism classes of) objects \(s\) in \(S\) under translation. For \(* = 0\) we obtain the \(K_0\) result. Reading this for \(* = 1\) yields \(K_1(S) = \pi_1(\Sigma S^{-1}S) = H_1(\text{B Aut}(s)) = \text{colim} H_1(B \text{ Aut}(s)) = K_1^\text{det}(S)\).

**Remark.** In [4], Bass defined groups \(K_2^\text{det}(S)\). In the next section we will show that this agrees with the \(K_2(S)\) of Quillen.

Another pleasing property is that the spaces \(\Sigma S^{-1}S\) are infinite loop spaces. This follows from the fact that \(\pi_0 BS^{-1}S\) is the group \(K_0^\text{det}(S)\) and Proposition 2 below. For example, \(B \text{ Ens}^{-1}\text{Ens}\) is the space \(\Omega^\infty S^\infty\), and \(B \text{ P}^{-1}\text{P}\) is the space \(K_0(\mathbb{R}) \times B \text{ Gl}(\mathbb{R})^+\) (see page 91 of [1]).

**Proposition 2.** If \(T\) is a small monoidal category, \(BT\) is a homotopy associative \(H\)-space. If \(T\) is symmetric monoidal, \(BT\) is also homotopy commutative, and \(BT\) is an infinite loop space if and only if \(\pi_0(BT)\) is an abelian group.

**Remark.** There is a simple, purely algebraic definition of \(\pi_0(BT)\). If \(T\) is a small symmetric monoidal category, define \(\pi_0(T)\) to be the set of objects of \(T\), modulo the equivalence relation generated by requiring \(s \sim t\) whenever there is an arrow from \(s\) to \(t\). The product \(\square\) makes \(\pi_0(T)\) an abelian monoid. If \(T\) is in **SMCat** \(\pi_0(T)\) is the monoid of isomorphism classes of objects. Since \(\pi_0(T) = \pi_0(BT)\), the topological space \(BT\) is an infinite loop space if \(\pi_0(T)\) is a group.
Proof. The functor \[ T \times T \to T \] induces \[ BT \times BT \cong B(T \times T) \to BT, \]
making \( BT \) an \( H \)-space. Associativity (and commutativity in the symmetric case) of \( \square \) up to natural equivalence translates directly into homotopy associativity (and commutativity) of \( BT \). To determine when \( BT \) is an infinite loop space, we use Segal’s machine [12]; this is appropriate since Thomason has shown in [13] that \( BT \) is the initial space of a \( \Gamma \)-space. By [15, p. 461], \( B\square \) has a homotopy inverse iff \( \pi_0(BT) \) is a group, and by [12] this is necessary and sufficient for \( BT \) to be an infinite loop space.

Remark. We could have also used May’s machine. In the relevant vocabulary, \( BT \) is an \( A_\infty \) space if \( T \) is monoidal, and \( BT \) is an \( E_\infty \) space if \( T \) is symmetric monoidal. This was shown in [9]. The above formulation of Proposition 2 was shown to me by Z. Fiedorowicz.

The usefulness of Proposition 2 is that some of the \( S \) in \( SMCat \) already have a group \( \pi_0S \). In this case, the natural map \( BS \to BS^{-1}S \) is a homotopy equivalence (it is an infinite loop space map which is a homology isomorphism). For example, this is true of \( S = \text{Pic} \). It follows from [2] or [3] that \( B \text{Pic} \cong \text{Pic}(R) \times BU(R) \), where \( \text{Pic}(R) \) is the Picard group of the commutative ring \( R \), and \( U(R) \) is the group of units of \( R \). We have the

Corollary. \( K_0 \text{Pic} = \text{Pic}(R) \), \( K_1 \text{Pic} = U(R) \), and the groups \( K_\ast \text{Pic} \) are zero for \( \ast > 2 \).

2. The plus construction and \( K_2 \). If the category \( S \) has a countable, cofinal subcategory, we can construct a group \( \text{Aut}(S) \) playing the role that \( G_1(Z) \) does for \( P \). The construction is given on page 355 of [3], although the constructions of [2, p. 25], [4, p. 197], and [14] may be used where appropriate.

The groups \( \text{Aut}(S) \) are easy to compute in the sample categories given in the introduction. The free modules in \( P \) and \( FP \) allow us to take \( \text{Aut}(P) = G_1(R) \), \( \text{Aut}(FP) = G_0(R) = \text{colim} \{ G_1(R); \alpha \mapsto \alpha \otimes 1 \} \). \( \text{Aut}(\text{Pic}) \) is just \( U(R) \). The matrix rings in \( Az \) allow us to take \( \text{Aut}(Az) \) to be the direct colimit of the \( R \)-algebra automorphisms of the \( M_n(R) \). We have \( \text{Aut}((\text{Quad}^A(A, \Lambda))) = U^\Lambda(A, \Lambda) = \text{colim} U^\Lambda_n(A, \Lambda) \) and \( \text{Aut}(\text{Ens}) = \Sigma = \text{colim} \Sigma_n \).

Proposition 3. Suppose that \( S \) has a countable, cofinal subcategory, so that \( \text{Aut}(S) \) exists. Then the commutator subgroup \( E \) of \( \text{Aut}(S) \) is a perfect, normal subgroup, so the plus construction may be applied to \( B \text{Aut}(S) \). The resulting space is the basepoint component of \( BS^{-1}S \), i.e., \( BS^{-1}S \cong K_0(S) \times B \text{Aut}(S)^+ \). Moreover, \( K_1(S) = \text{Aut}(S)/E \).

Proof. As \( E \) is a direct colimit, every element of \( E \) is a product of elements, each represented by a commutator \( [\alpha, \beta] \) in some \( \text{Aut}(S) \). We compute in \( \text{Aut}(S \sqcup S \sqcup S) \) that \( [\alpha, \beta] \sqcup 1 \sqcup 1 = [\alpha \sqcup \alpha^{-1} \sqcup 1, \beta \sqcup 1 \sqcup 1] \beta^{-1} \), which represents an element of \( [E, E] \) by the Abstract Whitehead Lemma on page 351 of [3]. This shows that \( E \) is perfect, so that \( f: B \text{Aut}(S) \to B \text{Aut}(S)^+ \) exists and is any acyclic map with \( \ker(\pi_1 f) = E \). If we copy the telescope construction of [6], we obtain such an acyclic map from \( B \text{Aut}(S) \) to the basepoint component of \( BS^{-1}S \), proving the proposition.
We are now in a position to compare Quillen’s $K_2$ to Bass’s $K_2^{\text{det}}$. In Appendix A to [4], Bass defined $K_2^{\text{det}}(S)$ to be the direct colimit of the groups $H_0(\text{Aut}(s); [\text{Aut}(s), \text{Aut}(s)])$.

We remark that when $\text{Aut}(S)$ exists we have $K_2^{\text{det}}(S) = H_2(E)$. This may be seen by reading the proof of (A.6) on page 200 of [4]. In this case, $K_2^{\text{det}}(S)$ may also be interpreted as the kernel of a universal central extension of the perfect group $E$ (as in [11]).

**Theorem 4.** Quillen’s $K_2(S)$ is the same as Bass’s $K_2^{\text{det}}(S)$.

**Proof.** Any $S$ in $\text{SMCat}$ is the direct colimit of full subcategories which are countable, and hence for which $\text{Aut}(S)$ exists. As Bass’s and Quillen’s groups both commute with direct colimits, we are reduced to proving the theorem when $\text{Aut}(S)$ exists. In this case we have to show that $K_2(S) = H_2(E)$. We will use a modification of the proof of Proposition 4.12 in [9], which is essentially due to D. W. Anderson.

There is a homotopy fibration $BE \to B\text{Aut}(S) \to B(K_1S)$. Since $K_1S$ is an abelian group, $B(K_1S)$ is an Eilenberg-Mac Lane space. The map $B\text{Aut}(S) \to B(K_1S)$ factors through an $H$-space map $B\text{Aut}(S)^+ \to B(K_1S)$ by universality of the plus construction. If $F$ denotes the fiber of the latter map, there is a map of fibrations:

\[
\begin{array}{ccc}
BE & \to & B\text{Aut}(S) & \to & B(K_1S) \\
\downarrow & & \downarrow & & \downarrow \\
F & \to & B\text{Aut}(S)^+ & \to & B(K_1S).
\end{array}
\]

The action of $K_1S = \pi_1(B(K_1S))$ on $BE$ is trivial for the same reasons given in [9]: if $y \in \text{Aut}(s)$ represents $[y] \in K_1S$ and $z \in H_* (BE)$, we can choose a subgroup $\text{Aut}(t)$ of $\text{Aut}(s \sqcup t)$ for some $t$ so that $z$ is in the image of $H_* (B[\text{Aut}(t), \text{Aut}(t)])$. As $y$ commutes with $\text{Aut}(t)$, $[y]$ acts trivially on $z$. On the other hand, the action of $K_1S$ on $H_* (F)$ is trivial because $F$ is connected and is the fiber of an $H$-space map (see [5, p. 16-09]). It follows by the Comparison Theorem (in [8]) that $H_* (E) = H_* (BE) \to H_* (F)$ is a homology isomorphism. On the other hand, $F$ is simply connected, so $H_3(F) \approx \pi_2(F) \approx \pi_2(B\text{Aut}(S)^+) = K_2(S)$.

We will need the following result which is implicit in [10, p. 96], and was pointed out in [14]. The proof involves a comparison of the groups $\text{Gl}(R)$ and $\text{Gl}_\Theta(R)$.

**Proposition 5.** $K_* (FP) = Q \otimes K_* (P) = Q \otimes K_* (R)$ for $* > 1$, while $K_0(FP) = U^+ (Q \otimes K_0(R))$ in the notation of [3, p. 516].

3. **Azumaya algebras.** In this section we compute the groups $K_* Az$. The computation was inspired by the calculations of [2] and [14]. I am indebted to C. McGibbon and J. Neisendorfer for suggesting the use of the Comparison Theorem in the proof.

There is a functor $\text{End}: FP \to Az$ in $\text{SMCat}$, which sends a faithful projective $R$-module $P$ to its endomorphism ring $\text{End}(P)$, and sends the automorphism $\alpha$ of $P$ to conjugation by $\alpha$. This induces a map $\text{End}: \text{Gl}_\Theta(R) \to \text{Aut}(Az)$. The following result is proven in [2] and on page 74 of [3]:

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Proposition 6 (Rosenberg-Zelinsky). There is an exact sequence

\[ 1 \to U(R) \to \text{End}(\text{Az}) \to \text{Aut}(\text{Az}) \to T \text{Pic}(R) \to 1, \]

where \( T \text{Pic}(R) \) is the torsion subgroup of \( \text{Pic}(R) \).

We consider \( T \text{Pic}(R) \) to represent outer automorphisms, and would like a category of inner automorphisms. We define \( \text{In} \) to be the image of \( \text{End} \). \( \text{In} \) is the monoidal subcategory of \( \text{Az} \) whose objects are the \( \text{End}(P) \), and whose arrows are “inner” automorphisms. The group \( \text{Aut}(\text{In}) = \text{colim} \text{Aut}(\text{M}_n(R)) \) is the group \( P\text{Gl}_\otimes(R) = \text{Gl}_\otimes(R)/U(R) \) (cf. pages 108, 119 of [2] and page 74 of [3]). We thus have short exact sequences of groups \( 1 \to U(R) \to \text{Gl}_\otimes(R) \to P\text{Gl}_\otimes(R) \to 1 \) and \( 1 \to P\text{Gl}_\otimes(R) \to \text{Aut}(\text{Az}) \to T \text{Pic}(R) \to 1 \). The sequence \( \text{Pic} \to \text{FP} \to \text{In} \) gives rise to a commutative diagram of spaces

\[
\begin{array}{ccc}
BU(R) & \to & B \text{Gl}_\otimes(R) \\
\downarrow & & \downarrow \\
B_0 \text{Pic}^{-1} \text{Pic} & \to & B_0 \text{FP}^{-1} \text{FP} \\
\alpha & & B_0 \text{In}^{-1} \text{In}.
\end{array}
\]

The top row is a fibration, and the bottom row is a sequence of infinite loop spaces and infinite loop maps. The left vertical arrow is a homotopy equivalence of infinite loop spaces by Proposition 2. As the bottom composite is trivial, there is an infinite loop space map from \( B_0 \text{Pic}^{-1} \text{Pic} \) to the fiber \( X \) of the lower right horizontal map \( \alpha \). Summarizing, there is a map of fibrations

\[
\begin{array}{ccc}
BU(R) & \to & B \text{Gl}_\otimes(R) \\
\downarrow & & \downarrow \\
X & \to & B_0 \text{FP}^{-1} \text{FP} \\
\to & & \to \\
B_0 \text{In}^{-1} \text{In}
\end{array}
\]

in which the map \( BU(R) \to X \) is an \( H \)-map. Now \( P\text{Gl}_\otimes(R) \) acts trivially on \( H_*(BU(R)) \) because \( U(R) \) is central in \( \text{Gl}_\otimes(R) \) (any element of \( P\text{Gl}_\otimes(R) \) induces the identity map on \( BU(R) \)). Moreover, \( \pi_1(B_0 \text{In}^{-1} \text{In}) \) acts trivially on \( H_*(X) \) because \( \alpha \) is an \( H \)-map and \( X \) is connected (see [5, p. 16-09]). Hence the Comparison Theorem [8, p. 355] applies: as the base and total space maps are homology isomorphisms (by Proposition 3), the infinite loop space map \( BU(R) \to X \) is a homology isomorphism, hence a homotopy equivalence. We have proven:

Theorem 7. \( BU(R) \to B \text{Gl}_\otimes(R)^+ \to B P\text{Gl}_\otimes(R)^+ \) is a homotopy fibration.

Corollary 8. For \( * > 3 \), \( K_\ast \text{In} \cong K_\ast \text{FP} \cong Q \otimes K_\ast(R) \). If \( \mu(R) \) denotes the roots of unity of \( R \),

\[ K_2 \text{In} = \mu(R) \oplus K_2 \text{FP} = \mu(R) \oplus (Q \otimes K_2(R)). \]

Finally, \( K_1 \text{In} = K_1 \text{FP} / \text{im} U(R) \) and \( K_0 \text{In} = U^+(Q \otimes K_0(R)) / \text{im}(\text{Pic}(R)) \).

Proof. Use the long exact homotopy sequence and the fact that \( \pi_\ast BU = 0 \) for \( \ast \neq 1 \), as well as Proposition 5. The only subtleties are that in the sequence \( 0 \to K_2 \text{FP} \to K_1 \text{In} \to U(R) \to K_1 \text{FP} \) the left map splits (by divisibility of \( K_2 \text{FP} \)) and that the kernel of the right map is the torsion subgroup \( \mu(R) \) of \( U(R) \).
Theorem 9. There is a long exact sequence in K-theory:

\[ \cdots \to K_{*+1}A_R \to K_*\text{Pic} \to K_*\text{FP} \to K_*A_R \to \cdots \]

In particular: for \( * > 3 \), \( K_*A_R = K_*\text{FP} = Q \otimes K_*(R) \), \( K_2A_R = \mu(R) \oplus K_2\text{FP} = \mu(R) \oplus (Q \otimes K_2(R)) \),

\[ K_1A_R = T \text{Pic}(R) \oplus (Q/Z \otimes U(R)) \oplus (Q \otimes SK_1(R)), \]

and \( K_0A_R = \text{Br}(R) \oplus U^+(Q \otimes K_0(R))/\text{im Pic}(R) \), where \( \text{Br}(R) \) is the Brauer group of \( R \).

Proof. The map \( B \text{Aut}(\text{In}) \to B \text{Aut}(\text{Az}) \) is (up to homotopy) a covering space map with fiber the abelian group \( T \text{Pic}(R) \). The commutator groups \( [\text{Aut}(\text{In}), \text{Aut}(\text{In})] \) and \( [\text{Aut}(\text{Az}), \text{Aut}(\text{Az})] \) are isomorphic. Hence we can perform a \( T \text{Pic} \)-equivariant plus construction on \( B \text{Aut}(\text{In}) \): for every cell we attach, all translates of the cell are also attached. In this way we obtain the model \( B \text{Aut}(\text{In})^+ / T \text{Pic} \) for \( B \text{Aut}(\text{Az})^+ \), and a fibration \( T \text{Pic}(R) \to B \text{Aut}(\text{In})^+ \to B \text{Aut}(\text{Az})^+ \). This yields \( K_*A_R \) for \( * > 2 \). Bass’s analysis of the low-dimensional terms in \([2]\) gives \( K_0, K_1 \) and a fibration \( B \text{Pic}^+ \text{Pic} \to B \text{FP}^+ \text{FP} \to B A^+A_R \).

Remark. We have shown that the commutator subgroup \( E \) of \( \text{PGL}_n(R) \) is perfect. In fact, it is the subgroup generated by the images of the elementary matrices in the \( \text{GL}_n(R) \), so the fact that \( E = [E, E] \) may be deduced from the fact that elementary matrices are commutators in \( \text{GL}_n \), \( n > 3 \). More interesting is the following consequence of Corollary 8: the torsion subgroup of \( H_2(E) \) is isomorphic to the roots of unity in the ring \( R \). It would be interesting to find an explicit description of this isomorphism, especially for \( R = \mathbb{C} \).

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