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A NEW PROOF OF A RESULT OF LEVITZKI

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ABSTRACT. A very short proof of the result which states that a nil ring of bounded index has a nonzero nilpotent ideal is given and it is shown that the same method of proof yields a more general result.

We present a new proof of the following well-known result which is attributed to Levitzki ([1, p. 374], [2]; for a proof see [3, pp. 1–2]): If a ring R has a nonzero right ideal A which is nil of bounded index, then R has a nonzero nilpotent ideal.

An element $b \in R$ satisfying $bRb = 0$ generates a nilpotent ideal, so we prove the existence of such a nonzero element.

Let $n > 1$ be such that $x^n = 0$ for all $x \in A$ and take $a \in A$, $a \neq 0$, $a^2 = 0$. It follows that $(ay)^{n-1}a = (ay + a)^n = 0$ for all $y \in R$. Let $k \geqslant 1$ be minimal for which there exists $0 \neq b \in R$ such that $(by)^k b = 0$ for all $y \in R$. If $k > 1$, take $y_0 \in R$ satisfying $(by_0)^{k-1}b \neq 0$ and let $b_1 = (by_0)^{k-1}b$. Since b_1 is a right multiple of b and $by_0b_1 = 0$ we get

$$0 = \left(b_1z + (by_0)^{k-1}\right)^k b = (b_1z)^{k-1}(by_0)^{k-1}b = (b_1z)^{k-1}b_1$$

for all $z \in R$. This contradicts the minimality of k, so $k = 1$ and $bRb = 0$.

We now show that the same arguments as before can be used to give a proof of a generalization of Levitzki's result which is given in [2]. It states that if a ring R has a nonzero right ideal A for which there exist $c \in R$ and a fixed $n > 1$ such that $Ac \neq 0$ and $x^n c = 0$ for all $x \in R$, then R has a nonzero nilpotent ideal.

We take $a \in A$, $ac \neq 0$, $a^2c = 0$ and we get $(acy)^{n-1}ac = (acy + a)^n c = 0$ for all $y \in R$. Then we let $k \geqslant 1$ be minimal for which there exists $b \in R$, $bc \neq 0$, $(bcy)^k bc = 0$ for all $y \in R$. The assumption $k > 1$ yields a contradiction as above, for if we take $y_0 \in R$ satisfying $(bcy_0)^{k-1}bc \neq 0$ and $b_1 = (bcy_0)^{k-1}b$, we get $b_1c \neq 0$ and

$$0 = \left(b_1cz + (bcy_0)^{k-1}\right)^k bc = (b_1cz)^{k-1}(bcy_0)^{k-1}bc = (b_1cz)^{k-1}b_1c$$

for all $z \in R$.

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