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ANOTHER GRADED ALGEBRA WITH A NONRATIONAL HILBERT SERIES

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ABSTRACT. Recently J. B. Shearer has constructed a graded algebra with a nonrational Hilbert series, a counterexample to Govorov's conjecture. In this note we give a simpler example.

Let $k\langle X_1, \ldots, X_r\rangle$ denote the free associative algebra in r variables $X_1, \ldots, X_r$ of degree 1 over a field k. Let I be a finitely-generated homogeneous ideal of $k\langle X_1, \ldots, X_r\rangle$. The quotient ring $R = k\langle X_1, \ldots, X_r\rangle/I$ is naturally a graded algebra over k. The Hilbert series of R is the formal power series

$$H_R(Z) = \sum_{n \geq 0} (\dim_k R_n) Z^n,$$

where R_n is the homogeneous component of R of degree n.

V. E. Govorov conjectured that $H_R(Z)$ is a rational function [2], [3]. This conjecture has been of interest in connection with the problem of the rationality of the Poincaré series of local rings (see for example [4], [5]). Recently, J. B. Shearer [6] has given a counterexample to the conjecture. He has constructed a monoid algebra with a nonrational Hilbert series, which is generated by 10 elements with 57 quadratic relations. In this note we give a simpler example, a monoid algebra with a nonrational Hilbert series generated by 6 elements with 9 quadratic relations.

Let F be a free monoid on a set $\{a, a^+, b, b^+, x, y\}$, and let $F^0 = F \cup \{0\}$ be the monoid F with a zero adjoined. We define a monoid M as a quotient of F^0 modulo the following relations (i)–(iii).

(i) $aa^+ = a^+a,\ bb^+ = b^+b,\ ab^+ = a^+b,\ ba^+ = b^+a.$

(ii) $ax = xa,\ bx = xb^+,\ a^+y = ya^+,\ by = yb^+.$

(iii) $xy = 0.$

Two words ρ and σ in F^0 are called equivalent, if they are transformed to each other by a finite series of substitutions (i)–(iii). Let M_n be the set of all nonzero classes of equivalent words of length n. Then

$$M = \bigcup_{n \geq 0} M_n \cup \{0\} \qquad \text{(disjoint union)}.$$

For a subset N of M, we define the formal power series $H_N(Z)$ called the Hilbert series of N as

$$H_N(Z) = \sum_{n \geq 0} |N \cap M_n| Z^n.$$

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Let $R = k[M]$ be the monoid algebra of M over k. Then we have

$$H_R(Z) = H_M(Z) = \sum_{n \geq 0} |M_n| Z^n.$$

We will prove that $H_M(Z)$ is not rational.

We consider the following 8 subsets of M.

A: the submonoid of M generated by a, a^+, b, b^+.

B (resp. B^+): the submonoid of M generated by a, b (resp. a^+, b^+),

X (resp. X^*): the set of all equivalence classes of words in a^+, b^+, x without a^+, b^+ in the last (resp. first) position,

Y (resp. Y^*): the set of all equivalence classes of words in a, b, y without a, b in the first (resp. last) position,

C: the set of all equivalence classes of words in the form of $x\sigma y$ ($\sigma \in A$).

Easily we find

$$H_A(Z) = 1/(1-2Z)^2, \quad H_B(Z) = H_{B^+}(Z) = 1/(1-2Z),$$
$$H_X(Z) = H_{X^*}(Z) = H_Y(Z) = H_{Y^*}(Z)$$
$$= 1 + Z/(1-3Z) = (1-2Z)/(1-3Z).$$

Let ρ be any word. By transferring every x contained in ρ to the left and y to the right as much as possible applying substitutions (i) and (ii), we get a word $\bar{\rho}$ equivalent to ρ whose every subword without x, y appearing just before x (resp. after y) consists of only a^+, b^+ (resp. a, b). This $\bar{\rho}$ takes the form of

$$\beta\phi_1\gamma_1\psi_1\phi_2\gamma_2\psi_2 \cdots \phi_s\gamma_s\psi_s\beta', \tag{1}$$

where $\phi_i \in X^*$, $\psi_i \in Y^*$, $\gamma_i \in C$ for $i = 1, \ldots, s$, and $\beta \in B^+$ or $\beta = \sigma\psi y$ ($\sigma \in A$, $\psi \in Y$) and $\beta' \in B$ or $\beta' = x\phi\sigma$ ($\sigma \in A$, $\phi \in X$) (s is possibly 0, that is, the subword between β and β' could be empty). Conversely, to every expression given as (1) a unique equivalence class of words corresponds. Therefore the Hilbert series $H_M(Z)$ is given as

$$H_M(Z) = \frac{(H_{B^+}(Z) + ZH_A(Z)H_Y(Z))(H_B(Z) + ZH_A(Z)H_X(Z))}{1 - H_{X^*}(Z)H_C(Z)H_{Y^*}(Z)}$$
$$= \frac{1}{(1-3Z)^2 - (1-2Z)^2 H_C(Z)}.$$

Thus the proof of the irrationality of $H_M(Z)$ is reduced to that of $H_C(Z)$.

Define $d(n) = |\{\sigma \in A \cap M_n | x\sigma y = 0 \text{ in } M\}|$ for $n \geq 0$ and $D(Z) = \sum_{n \geq 0} d(n)Z^n$. Then we have $H_C(Z) = Z^2(H_A(Z) - D(Z))$. We will show that $D(Z)$ is not rational.

Let $\sigma \in A$. Moving the y in the last position of $x\sigma y$ leftward as much as possible using substitutions (i) and (ii), we get a word $x\sigma_1 y\sigma_2$ equivalent to $x\sigma y$ such that $\sigma_2 \in B^+$ and $\sigma_1 = 1$ or

$$\sigma_1 = \tau_1 a \qquad (\tau_1 \in B) \tag{2}$$

or

$$\sigma_1 = \tau_1 b^+ \qquad (\tau_1 \in B^+). \tag{3}$$

If $\sigma_1 = 1$, then $x\sigma y = 0$. Let us consider case (2) (the argument on case (3) is parallel). For a word $\sigma \in A$, $\alpha(\sigma)$ (resp. $\beta(\sigma)$) denotes the number of a's (resp. b's) contained in the string of a and b obtained by dropping all $+$'s of σ. If the word σ_2 has a left divisor $\sigma_3 \in B^+$ ($\sigma_2 = \sigma_3\sigma_4$ for some $\sigma_4 \in B^+$) such that $\alpha(\sigma_3) - \beta(\sigma_3) = \beta(\sigma_1)$, then

$$x\sigma y = x\sigma_1 y\sigma_2 = x\sigma_1\bar{\sigma}_3 y\sigma_4 = \sigma_1\tilde{\sigma}_3 xy\sigma_4 = 0,$$

where $\bar{\sigma}_3$ (resp. $\tilde{\sigma}_3$) is the word obtained from σ_3 by changing every b^+ in σ_3 to b (resp. by dropping all $+$'s in σ_3). Conversely, $x\sigma y = 0$ happens only under this situation.

Let $d(n, r)$ for $n, r \geqslant 0$ be the number of words $\sigma \in B$ of length n having left divisors $\sigma' \in B$ such that $\alpha(\sigma') - \beta(\sigma') = r$. Then the preceding argument gives

$$d(n) = 2^n + 2 \cdot \sum_{r>0} \sum_{l+m=n-1} \binom{l}{r} d(m, r). \tag{4}$$

Let us define $D_r(Z) = \Sigma_{n \geqslant 0} d(n, r)Z^n$. Using the formula

$$\sum_{n \geqslant 0} \binom{n}{r} Z^n = \frac{Z^r}{(1 - Z)^{r+1}},$$

we obtain from (4) that

$$D(Z) = \frac{1}{1 - 2Z} + \frac{2Z}{1 - Z} \cdot \sum_{r>0} \left(\frac{Z}{1 - Z}\right)^r D_r(Z). \tag{5}$$

Clearly we have

$$d(n, 0) = 2^n \quad \text{and} \quad d(n, r) = 0 \quad \text{for } n < r. \tag{6}$$

In particular

$$D_0(Z) = 1/(1 - 2Z). \tag{7}$$

For $n, r \geqslant 1$ we see

$$d(n, r) = d(n - 1, r - 1) + d(n - 1, r + 1). \tag{8}$$

It follows from (6) and (8) that

$$D_r(Z) = Z(D_{r-1}(Z) + D_{r+1}(Z)) \tag{9}$$

for $r \geqslant 1$. Let $e(n)$ be the number of sequences $\varepsilon_1, \ldots, \varepsilon_n$ consisting of 1 and -1 of length n such that $1 + \varepsilon_1 + \cdots + \varepsilon_n = 0$ and $\varepsilon_1 + \cdots + \varepsilon_i \geqslant 0$ for all $i < n$. We define $e(0) = 0$ for convenience. Now define

$$E(Z) = \sum_{n \geqslant 0} e(n)Z^n.$$

From (9) we obtain

$$D_1(Z) = D_0(Z)E(Z). \tag{10}$$

Clearly $e(1) = 1$. From the definition of $e(n)$ we see

$$e(n) = \sum_{l+m=n-1} e(l)e(m)$$

for $n \geqslant 2$. Therefore we have

$$E(Z) = Z + Z(E(Z))^2. \tag{11}$$

This means $E(Z)$ is not rational (but algebraic). (11) also means that $E(Z)$ satisfies the characteristic equation $ZX^2 - X + Z = 0$ of the linear difference equation (9). Hence observing (7) and (10) we get

$$D_r(Z) = D_0(Z)(E(Z))^r = (E(Z))^r/(1 - 2Z).$$

Therefore by (5) we obtain

$$\begin{aligned} D(Z) &= \frac{1}{1-2Z} + \frac{2Z}{(1-2Z)(1-Z)} \cdot \sum_{r>0} \left(\frac{ZE(Z)}{1-Z} \right)^r \\ &= \frac{1}{1-2Z}\left(1 + \frac{2Z}{1-Z-ZE(Z)}\right). \end{aligned}$$

This shows that $D(Z)$ is not rational. Consequently $H_M(Z)$ is not rational either.

REMARK 1. The number of the relations cannot be reduced to 1, because a single relation always gives rise to a rational Hilbert series by Backelin [**1**].

REMARK 2. Recently Ufnarovsky [**7**] has shown that a graded algebra with a nonrational Hilbert series is constructed as a universal enveloping algebra of a certain Lie algebra. But there a generator of degree greater than 1 is needed. See also [**6**, Added in proof].

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