ANOTHER GRADED ALGEBRA
WITH A NONRATIONAL HILBERT SERIES

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Abstract. Recently J. B. Shearer has constructed a graded algebra with a nonrational Hilbert series, a counterexample to Govorov's conjecture. In this note we give a simpler example.

Let $k\langle X_1, \ldots, X_r \rangle$ denote the free associative algebra in $r$ variables $X_1, \ldots, X_r$ of degree 1 over a field $k$. Let $I$ be a finitely-generated homogeneous ideal of $k\langle X_1, \ldots, X_r \rangle$. The quotient ring $R = k\langle X_1, \ldots, X_r \rangle/I$ is naturally a graded algebra over $k$. The Hilbert series of $R$ is the formal power series

$$H_R(Z) = \sum_{n>0} (\dim_k R_n)Z^n,$$

where $R_n$ is the homogeneous component of $R$ of degree $n$.

V. E. Govorov conjectured that $H_R(Z)$ is a rational function [2], [3]. This conjecture has been of interest in connection with the problem of the rationality of the Poincaré series of local rings (see for example [4], [5]). Recently, J. B. Shearer [6] has given a counterexample to the conjecture. He has constructed a monoid algebra with a nonrational Hilbert series, which is generated by 10 elements with 57 quadratic relations. In this note we give a simpler example, a monoid algebra with a nonrational Hilbert series generated by 6 elements with 9 quadratic relations.

Let $F$ be a free monoid on a set $\{a, a^+, b, b^+, x, y\}$, and let $F^0 = F \cup \{0\}$ be the monoid $F$ with a zero adjoined. We define a monoid $M$ as a quotient of $F^0$ modulo the following relations (i)--(iii).

(i) $aa^+ = a^+a, bb^+ = b^+b, ab^+ = a^+b, ba^+ = b^+a$.
(ii) $ax = xa, bx = xb^+, a+y = ya^+, by = yb^+$.
(iii) $xy = 0$.

Two words $\rho$ and $\sigma$ in $F^0$ are called equivalent, if they are transformed to each other by a finite series of substitutions (i)--(iii). Let $M_n$ be the set of all nonzero classes of equivalent words of length $n$. Then

$$M = \bigcup_{n>0} M_n \cup \{0\} \quad \text{(disjoint union)}.$$

For a subset $N$ of $M$, we define the formal power series $H_N(Z)$ called the Hilbert series of $N$ as

$$H_N(Z) = \sum_{n>0} |N \cap M_n|Z^n.$$

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Let $R = k[M]$ be the monoid algebra of $M$ over $k$. Then we have

$$H_R(Z) = H_M(Z) = \sum_{n>0} |M_n| Z^n.$$ 

We will prove that $H_M(Z)$ is not rational.

We consider the following 8 subsets of $M$.

$A$: the submonoid of $M$ generated by $a, a^+, b, b^+$.

$B$ (resp. $B^+$): the submonoid of $M$ generated by $a, b$ (resp. $a^+, b^+$),

$X$ (resp. $X^*$): the set of all equivalence classes of words in $a^+, b^+, x$ without $a^+, b^+$ in the last (resp. first) position,

$Y$ (resp. $Y^*$): the set of all equivalence classes of words in $a, b, y$ without $a, b$ in the first (resp. last) position,

$C$: the set of all equivalence classes of words in the form of $xay$ ($\sigma \in A$).

Easily we find

$$H_A(Z) = \frac{1}{(1 - 2Z)^2}, \quad H_B(Z) = H_B^*(Z) = \frac{1}{1 - 2Z},$$

$$H_X(Z) = H_X^*(Z) = H_Y(Z) = H_Y^*(Z) = \frac{1 + Z}{1 - 3Z} = \frac{(1 - 2Z)}{1 - 3Z}.$$ 

Let $p$ be any word. By transferring every $x$ contained in $p$ to the left and $y$ to the right as much as possible applying substitutions (i) and (ii), we get a word $\bar{p}$ equivalent to $p$ whose every subword without $x, y$ appearing just before $x$ (resp. after $y$) consists of only $a^+, b^+$ (resp. $a, b$). This $\bar{p}$ takes the form of

$$\beta\phi_1\gamma_1\psi_1\phi_2\gamma_2\psi_2 \cdots \phi_s\gamma_s\psi_s\beta',$$

where $\beta_i \in X^*, \gamma_i \in Y^*$, $\gamma_i \in C$ for $i = 1, \ldots, s$, and $\beta \in B^+$ or $\beta = \sigma\psi\gamma$ ($\sigma \in A, \psi \in Y$) and $\beta' \in B$ or $\beta' = x\phi\sigma$ ($\sigma \in A, \phi \in X$) ($s$ is possibly 0, that is, the subword between $\beta$ and $\beta'$ could be empty). Conversely, to every expression given as (1) a unique equivalence class of words corresponds. Therefore the Hilbert series $H_M(Z)$ is given as

$$H_M(Z) = \frac{(H_B^*(Z) + ZH_A(Z)H_Y(Z))(H_B(Z) + ZH_A(Z)H_X(Z))}{1 - H_{X^*}(Z)H_C(Z)H_{Y^*}(Z)} = \frac{1}{(1 - 3Z)^2 - (1 - 2Z)^2H_C(Z)}.$$ 

Thus the proof of the irrationality of $H_M(Z)$ is reduced to that of $H_C(Z)$.

Define $d(n) = |\{ \sigma \in A \cap M_n | x\sigma y = 0 \text{ in } M \}|$ for $n > 0$ and $D(Z) = \Sigma_{n>0}d(n)Z^n$. Then we have $H_C(Z) = Z^2(H_A(Z) - D(Z))$. We will show that $D(Z)$ is not rational.

Let $\sigma \in A$. Moving the $y$ in the last position of $x\sigma y$ leftward as much as possible using substitutions (i) and (ii), we get a word $x\sigma_1\gamma\sigma_2$ equivalent to $x\sigma y$ such that $\sigma_2 \in B^+$ and $\sigma_1 = 1$ or

$$\sigma_1 = \tau_1 a \quad (\tau_1 \in B) \quad (2)$$

or

$$\sigma_1 = \tau_1 b^+ \quad (\tau_1 \in B^+). \quad (3)$$
If $\sigma_1 = 1$, then $x\sigma y = 0$. Let us consider case (2) (the argument on case (3) is parallel). For a word $\sigma \in \mathcal{A}$, $\alpha(\sigma)$ (resp. $\beta(\sigma)$) denotes the number of $a$’s (resp. $b$’s) contained in the string of $a$ and $b$ obtained by dropping all $+$’s of $\sigma$. If the word $\sigma_2$ has a left divisor $\sigma_3 \in \mathcal{B}^+$ ($\sigma_2 = \sigma_3\sigma_4$ for some $\sigma_4 \in \mathcal{B}^+$) such that $\alpha(\sigma_3) - \beta(\sigma_3) = \beta(\sigma_1)$, then

$$x\sigma y = x\sigma_1y\sigma_2 = x\sigma_1\tilde{\sigma}_3y\sigma_4 = \sigma_1\tilde{\sigma}_3x\sigma_4 = 0,$$

where $\tilde{\sigma}_3$ (resp. $\tilde{\sigma}_3$) is the word obtained from $\sigma_3$ by changing every $b^+$ in $\sigma_3$ to $b$ (resp. by dropping all $+$’s in $\sigma_3$). Conversely, $x\sigma y = 0$ happens only under this situation.

Let $d(n, r)$ for $n, r > 0$ be the number of words $\sigma \in \mathcal{B}$ of length $n$ having left divisors $\sigma' \in \mathcal{B}$ such that $\alpha(\sigma') - \beta(\sigma') = r$. Then the preceding argument gives

$$d(n) = 2^n + 2 \cdot \sum_{r \geq 0} \sum_{l + m = n - 1} \binom{l}{r} d(m, r). \quad (4)$$

Let us define $D_r(Z) = \sum_{n \geq 0} d(n, r)Z^n$. Using the formula

$$\sum_{n \geq 0} \binom{n}{r} Z^n = \frac{Z^r}{(1 - Z)^{r+1}},$$

we obtain from (4) that

$$D_r(Z) = \frac{1}{1 - 2Z} + \frac{2Z}{1 - Z} \cdot \sum_{r \geq 0} \left( \frac{Z}{1 - Z} \right)^r D_r(Z). \quad (5)$$

Clearly we have

$$d(n, 0) = 2^n \quad \text{and} \quad d(n, r) = 0 \quad \text{for} \quad n < r. \quad (6)$$

In particular

$$D_0(Z) = \frac{1}{1 - 2Z}. \quad (7)$$

For $n, r > 1$ we see

$$d(n, r) = d(n - 1, r - 1) + d(n - 1, r + 1). \quad (8)$$

It follows from (6) and (8) that

$$D_r(Z) = Z(D_{r-1}(Z) + D_{r+1}(Z)) \quad (9)$$

for $r > 1$. Let $e(n)$ be the number of sequences $e_1, \ldots, e_n$ consisting of 1 and $-1$ of length $n$ such that $1 + e_1 + \cdots + e_n = 0$ and $e_1 + \cdots + e_i > 0$ for all $i < n$. We define $e(0) = 0$ for convenience. Now define

$$E(Z) = \sum_{n \geq 0} e(n)Z^n.$$

From (9) we obtain

$$D_1(Z) = D_0(Z)E(Z). \quad (10)$$

Clearly $e(1) = 1$. From the definition of $e(n)$ we see

$$e(n) = \sum_{l + m = n - 1} e(l)e(m)$$

for $n > 2$. Therefore we have

$$E(Z) = Z + Z(E(Z))^2. \quad (11)$$
This means $E(Z)$ is not rational (but algebraic). (11) also means that $E(Z)$ satisfies the characteristic equation $ZX^2 - X + Z = 0$ of the linear difference equation (9). Hence observing (7) and (10) we get

$$D_r(Z) = D_0(Z)(E(Z))^r = (E(Z))^r / (1 - 2Z).$$

Therefore by (5) we obtain

$$D(Z) = \frac{1}{1 - 2Z} + \frac{2Z}{(1 - 2Z)(1 - Z)} \cdot \sum_{r > 0} \left( \frac{ZE(Z)}{1 - Z} \right)^r$$

$$= \frac{1}{1 - 2Z} \left( 1 + \frac{2Z}{1 - Z - ZE(Z)} \right).$$

This shows that $D(Z)$ is not rational. Consequently $H_M(Z)$ is not rational either.

**Remark 1.** The number of the relations cannot be reduced to 1, because a single relation always gives rise to a rational Hilbert series by Backelin [1].

**Remark 2.** Recently Ufnarovsky [7] has shown that a graded algebra with a nonrational Hilbert series is constructed as a universal enveloping algebra of a certain Lie algebra. But there a generator of degree greater than 1 is needed. See also [6, Added in proof].

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**References**