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RATIONALLY VARYING POLARIZING SUBALGEBRAS IN NILPOTENT LIE ALGEBRAS

LAWRENCE CORWIN AND FREDERICK P. GREENLEAF[1]

ABSTRACT. Let $\mathfrak{N}$ be a nilpotent Lie algebra, with (vector space) dual $\mathfrak{N}^*$. We construct a map $l \mapsto \mathfrak{M}_l$ from a Zariski-open subset of $\mathfrak{N}^*$ to the set of subalgebras of $\mathfrak{N}$ such that $\mathfrak{M}_l$ varies rationally with l, $\mathfrak{M}_l$ is polarizing for l, $(\operatorname{Ad} x)\mathfrak{M}_l = \mathfrak{M}_{l'}$ ($l' = \operatorname{Ad}^* x \cdot l$) for all $x \in N$, and $\mathfrak{H}_\infty(l) \subseteq \mathfrak{M}_l \subseteq \mathfrak{K}_\infty(l)$, where $\mathfrak{H}_\infty(l)$ and $\mathfrak{K}_\infty(l)$ are the canonical subalgebras introduced by R. Penney.

1. Let N be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{N}$; for $l \in \mathfrak{N}^*$, define the bilinear form B_l by $B_l(X, Y) = l([X, Y])$. In the Kirillov theory of the unitary representations of N, an important role is played by a *polarizing* (or *maximal subordinate*) subalgebra of l–that is, a subalgebra $\mathfrak{M}_l$ which is a maximal isotropic subspace for B_l. As Kirillov [**4**] showed, such a subalgebra always exists; in general, it is not unique. For Kirillov's purposes, the lack of uniqueness was unimportant. He showed that if one defines χ_l on $M_l = \exp \mathfrak{M}_l$ by $\chi_l(\exp X) = \exp 2\pi i l(X)$ and induces χ_l to a representation π_l on N, then π_l is independent (up to equivalence) of the choice of $\mathfrak{M}_l$; moreover, π_l is irreducible, and every irreducible (unitary) representation of N is equivalent to some π_l. Indeed, the unitary dual $\hat{N}$ of N is homeomorphic to the orbit space of $\mathfrak{N}^*$ mod the coadjoint action of N; see [**1**].

For some purposes, it is useful to find concrete realizations of "most" of the irreducible representations $\pi_l \in N\hat{}$ which vary smoothly with l. Here, "most" means "for all l in a Zariski-open set of $\mathfrak{N}^*$"; see [**6**, p. 55ff] for a discussion. This problem has arisen in discussions of applications of representation theory to partial differential equations; it also arises when one attempts to characterize the Fourier transform of Schwartz class functions on nilpotent Lie groups. (See, e.g., [**2**] and [**7**]; in these cases, one was able to avoid serious problems because of the nature of the groups under discussion.)

In [**8**], [**9**] M. Vergne showed that there is a natural construction which yields polarizing subalgebras $\mathfrak{M}_l$ which vary rationally in l. (Her construction actually works for solvable Lie algebras, where additional conditions are imposed on the $\mathfrak{M}_l$.) For our purposes we seek polarizations which not only vary rationally in l, and covariantly under $\operatorname{Ad}^*(N)$, but which are also related to the canonical

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subalgebras $\mathfrak{H}_\infty(l) \subseteq \mathfrak{K}_\infty(l)$ introduced by Penney [5], and used to advantage in [3]. We show that the canonical objects $\mathfrak{H}_\infty(l) \subseteq \mathfrak{K}_\infty(l)$ all depend rationally on l, and that there is a rational choice of polarization that fits between them: $\mathfrak{H}_\infty(l) \subseteq \mathfrak{M}_l \subseteq \mathfrak{K}_\infty(l)$ for all generic l. An example shows that the polarizations described in [8], [9] do not always satisfy this condition.

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2. We begin with some generalities. Let E be a finite-dimensional real vector space, and let $U \subseteq E$ be Zariski-open. Suppose that W is some other real finite-dimensional vector space. We say that the field of vectors φ: $U \to W$ is rational if the coefficients of φ (with respect to any fixed bases of E, W) are rational functions of $v \in U$. A field of subspaces $\{W_v: v \in U\}$ is rational if there are rational fields $\varphi_1, \ldots, \varphi_k$: $U \to W$ which form a basis for W_v on a Zariski-open subset of U. Note that a field of linear operators T: $E \to \operatorname{Hom}(V, W)$ may be regarded as a field of vectors.

In what follows, E will always be the indexing vector space. We shall often suppress mention of the Zariski-open set U; in fact, U may vary somewhat during the discussion. We use "generic" to mean "Zariski-open". We shall use, e.g., W_v and $W(v)$ interchangeably to refer to the value at v of the field $\{W_v\}$ of subspaces.

2.1. LEMMA. *Let $\varphi_1, \ldots, \varphi_j$ be rational vector fields on W which are linearly independent at a point $v_0 \in E$. Then they are generically linearly independent, and there are rational vector fields $\varphi_{j+1}, \ldots, \varphi_n$ such that $\{\varphi_1(v), \ldots, \varphi_n(v)\}$ is a basis of W for generic $v \in E$.*

PROOF. Extend $\varphi_1(v_0), \ldots, \varphi_j(v_0)$ to a basis $\varphi_1(v_0), \ldots, \varphi_j(v_0), w_{j+1}, \ldots, w_n$, and define $\varphi_i(v) = w_i$ for $j + 1 \leqslant i \leqslant n$. It suffices to show that $\{\varphi_1(v), \ldots, \varphi_n(v)\}$ is a basis for generic v, and this is clear from the fact that $\operatorname{Det}(\varphi_1, \ldots, \varphi_n)$ is rational and not identically 0.

2.2. LEMMA. *Let $\varphi_1, \ldots, \varphi_n$ be rational vector fields on W which form a basis for generic v. Define vector fields $\psi_1, \ldots, \psi_n$ on W^* such that for generic v, $\psi_1(v), \ldots, \psi_n(v)$ is the dual basis to $\varphi_1(v), \ldots, \varphi_n(v)$. Then $\psi_1, \ldots, \psi_n$ are rational.*

PROOF. This is a simple application of Cramer's rule.

2.3. LEMMA. *Let $\{W_v\}$ be a rational field of subspaces in W. Then $\{W_v^\perp\}$ is a rational field of subspaces in W^*.*

PROOF. Let $\varphi_1, \ldots, \varphi_k$ be the rational fields which give a basis for the W_v at generic v. Extend these to a generic basis $\{\varphi_1, \ldots, \varphi_n\}$ of W, as in Lemma 2.1, and let $\{\psi_1, \ldots, \psi_n\}$ be the dual basis of fields in W^*, rational by Lemma 2.2. Then $W_v^\perp = \operatorname{span}(\psi_{k+1}(v), \ldots, \psi_n(v))$ for generic v, and the lemma is proved.

2.4. PROPOSITION. *Let T: $E \to \operatorname{Hom}(W_1, W_2)$ be a rational field of linear operators. Then*

(a) T^*: $E \to \operatorname{Hom}(W_2^*, W_1^*)$, *where* $T^*(v) = (T(v))^*$, *is rational*;

(b) Range T *is a rational field of subspaces in* W_2;

(c) Ker T *is a rational field of subspaces in* W_1;
(d) Rank T *takes on its maximum value generically*;
(e) Null(T) *takes on its minimum value generically.*

PROOF. For (a), choose bases in W_1 and W_2, and use the dual bases in W_2 and W_1; as the matrix for T^* is the transpose of the matrix for T, the result follows. Next, (c) and (e) follow from (b) and (d), plus the familiar formula Ker $T =$ (Range $T^*)^\perp$ and Lemma 2.3.

So it suffices to prove (b) and (d). For (b), choose v_0 such that $T(v_0)$ has maximal rank; choose vectors $w_1, \ldots, w_k \in W_1$ such that $\{T(v_0)w_j\}$ is a basis for Range $T(v_0)$. Then $U = \{v = \{T(v)w_j\colon 1 \leqslant j \leqslant k\}$ is linearly independent$\}$ is Zariski-open, and for $v \in U$, the fields $\varphi_j(v) = T(v)w_j$ span Range $T(v)$ (since Rank $T(v_0)$ is maximal). Thus Range T is rational. For (d), note that the set where Rank T is maximal is a union of sets like U.

3. In dealing with fields of subalgebras, we shall consider two situations: first, a fixed nilpotent Lie algebra $\mathfrak{N}$, along with a rational field $\{\mathfrak{N}_v\}$ of subalgebras; and second, a fixed space $V = \mathbf{R}^k$, along with Lie algebra structures depending rationally on a parameter $v \in E$. (That is, the structure constants with respect to a fixed basis are rational functions of v.) It is convenient to subsume the first case under the second. Let $\varphi_1, \ldots, \varphi_k$ be rational vector fields which (generically) give a basis for the $\mathfrak{N}_v$; for generic v,

$$[\varphi_i(v), \varphi_j(v)] = \sum_{h=1}^{k} c_{hij}(v)\varphi_h(v), \qquad 1 \leqslant h, i, j \leqslant k,$$

where the c_{hij} are rational functions of v. Now fix a basis $e_1, \ldots, e_k$ of $\mathbf{R}^k$, and define

$$[e_i, e_j]_v = \sum_{h=1}^{k} c_{hij}(v)e_h;$$

we obtain a rational field of Lie algebra structures on $\mathbf{R}^k$ isomorphic (in an obvious sense) to the field $\{\mathfrak{N}_v\}$.

If $\mathfrak{N}$ is a Lie algebra and $\mathfrak{I}$ is an ideal, we define $\mathfrak{Z}_{\mathfrak{I}}(\mathfrak{N}) = \{X \in \mathfrak{N}\colon [X, \mathfrak{N}] \subseteq \mathfrak{I}\}$; in particular, $\mathfrak{Z}_{(0)}(\mathfrak{N}) = \mathfrak{Z}(\mathfrak{N})$, the center of $\mathfrak{N}$. If $l \in \mathfrak{N}^*$, then we define

$$\mathfrak{R}_l = \text{radical of } B_l = \{X \in \mathfrak{N}\colon l([X, \mathfrak{N}]) = 0\},$$

$$\mathfrak{H}_l(\mathfrak{N}) = \mathfrak{H}_l = \mathfrak{H}_l^{(1)} = \text{ideal generated by } \mathfrak{R}_l,$$

$$\mathfrak{H}_l^{(j+1)} = \mathfrak{H}_{l|\mathfrak{H}^{(j)}}(\mathfrak{H}_l^{(j)}).$$

(See [5] for a discussion of the $\mathfrak{H}^{(j)}$.) We shall also write $\mathfrak{R}(l)$, etc., for typographical convenience.

3.1. THEOREM. *Let* $\mathfrak{N}$ *be a nilpotent Lie algebra,* $\{\mathfrak{N}_v\colon v \in E\}$ *a rational field of subalgebras,* $\{\mathfrak{I}_v\colon v \in E\}$ *a rational field of subalgebras such that* $\mathfrak{I}_v$ *is an ideal of* $\mathfrak{N}_v$ *for generic* v, *and* $\{l_v\colon v \in E\}$ *a rational field of elements of* $\mathfrak{N}^*$. *We regard* l_v *as restricted to* $\mathfrak{N}_v$. *Then the following fields are rational*:

(a) $\mathfrak{Z}_{\mathfrak{J}_v}(\mathfrak{N}_v)$;
(b) $\mathfrak{R}_{l_v}$ $(\subseteq \mathfrak{N}_v)$;
(c) $\mathfrak{H}_{l_v}^{(j)}, j = 1, 2, \ldots$.

PROOF. Let k be the generic dimension of $\mathfrak{N}_\alpha$; let $\{e_1, \ldots, e_k\}$ be the standard basis of $\mathbf{R}^k$. As noted previously, we may regard the $\mathfrak{N}_v$ as modeled on $\mathbf{R}^k$; we may also assume that (generically) $\mathfrak{J}_v = \operatorname{span}(e_1, \ldots, e_s)$.

(a) Let $[X, e_i]_v = \sum_{j=1}^k c_{ij}(v; X)e_j$, $X \in \mathbf{R}^k$. Then $\mathfrak{Z}_{\mathfrak{J}_v}(\mathfrak{N}_v)$ is the kernel of the map T_v: $\mathbf{R}^k \to \mathbf{R}^{k(k-s)}$ defined by

$$T_v(X) = \big(c_{ij}(v; X)\colon 1 \leqslant i \leqslant k, s < j \leqslant k\big).$$

(b) $\mathfrak{R}_{l_v}$ is the kernel of S_v: $\mathbf{R}^k \to \mathbf{R}^k$, defined by

$$S(X) = \big(l_v([e_1, X]_v), \ldots, l_v([e_k, X]_v)\big).$$

In both (a) and (b), we may now apply Proposition 2.4.

(c) It suffices to prove the result for the case $j = 1$, and then to apply induction. We may assume that $s = \dim \mathfrak{R}(l_v)$ is constant. Let $\varphi_1, \ldots, \varphi_s$ be rational vector fields giving bases for generic v. Choose v_0 with $\dim \mathfrak{H}(l_{v_0})$ maximal, and choose indices $i_1, \ldots, i_p, j_1, \ldots, j_p$ $(1 \leqslant p \leqslant r)$, such that $\{\varphi_1(v_0), \ldots, \varphi_s(v_0), [e_{i_1}, \varphi_{j_1}(v_0)], \ldots, [e_{i_r}, \varphi_{j_r}(v_0)]\}$ is a basis for $\mathfrak{H}(l_{v_0})$. Then the φ_j and ψ_p (where $\psi_p(v) = [e_{i_p}, \varphi_{j_p}(v)]$) are rational vector fields giving a basis for $\mathfrak{H}(l_v)$ for generic v.

Recall (from **[5]**; see also **[3]**) that for fixed v, the $\mathfrak{H}_{l_v}^{(j)}$ are constant for large j (for $j \geqslant \dim n$, say); we denote this fixed algebra by $\mathfrak{H}^\infty(l_v)$. Its annihilator for the form B_{l_v} is denoted by $\mathfrak{K}^\infty(l_v)$; $\mathfrak{H}^\infty(l_v)$ and $\mathfrak{K}^\infty(l_v)$ are both subalgebras and $\mathfrak{H}^\infty(l_v)$ is the radical of $l_v|\mathfrak{K}^\infty(l_v)$. Furthermore, $\mathfrak{H}^\infty(l_v)$ is an ideal in $\mathfrak{K}^\infty(l_v)$, and any subalgebra of $l_v|\mathfrak{K}^\infty(l_v)$ also polarizes l_v (on $\mathfrak{N}_v$).

3.2. COROLLARY. *$\mathfrak{H}^\infty(l_v)$ and $\mathfrak{K}^\infty(l_v)$ depend rationally on v.*

PROOF. The first half is the case $j = n$ of Theorem 3.1(c). For the second, let $\{\varphi_j = 1 \leqslant j \leqslant r\}$ be a field of vectors which generically gives a basis for $\mathfrak{H}^\infty(l_v)$. Then $\mathfrak{K}^\infty(l_v) = \ker T_v$, where

$$T_v(X) = \big(l([\varphi_1(v), X]), \ldots, l([\varphi_r(v), X])\big).$$

4. The main results of this paper are consequences of the following lemma.

4.1. LEMMA. *Let $\{\mathfrak{N}_v\}$, $\{\mathfrak{J}_v\}$ be rational fields of subalgebras of a nilpotent Lie algebra $\mathfrak{N}$ such that $\mathfrak{J}_v$ is (generically) an ideal of $\mathfrak{N}_v$; let $\{l_v\}$ be a rational field of elements of $\mathfrak{N}^*$ such that $\mathfrak{J}_v$ is (generically) the radical of $B_{l_v}|\mathfrak{N}_v \times \mathfrak{N}_v$. Then there is a rational field $\{\mathfrak{M}_v\}$ of subalgebras such that (generically) $\mathfrak{M}_v$ polarizes $l_v|\mathfrak{N}_v$.*

PROOF. We may assume that the $\mathfrak{N}_v$ all have the same dimension, as do the $\mathfrak{J}_v$; let $\dim(\mathfrak{N}_v) - \dim \mathfrak{J}_v = 2m$. The proof is by induction on m, the case $m = 0$ being trivial. If $m > 0$, we use Theorem 3.1(a) to pick a rational vector field φ such that

$$\varphi(l_v) \in \mathfrak{Z}_{\mathfrak{J}(l_v)}(\mathfrak{N}(l_v)), \qquad \varphi(l_v) \notin \mathfrak{J}(l_v).$$

Let $\mathfrak{Z}_1(l_v) = \operatorname{span}(\mathfrak{Z}(l_v), \varphi(l_v))$, and let $\mathfrak{N}_1(l_v) = \mathfrak{Z}_1(l_v)^\perp$ (relative to B_{l_v} on $\mathfrak{N}_v$). Then $\mathfrak{Z}_1(l_v) \lhd \mathfrak{N}(l_v)$, and *a fortiori* $\mathfrak{Z}_1(l_v) \lhd \mathfrak{N}_1(l_v)$; also, $\mathfrak{Z}_1(l_v)$ is the radical of $B_{l_v}|\mathfrak{N}_1(l_v) \times \mathfrak{N}_1(l_v)$, as a dimension-counting argument shows. Clearly $\mathfrak{Z}_1(l_v)$ is rational, and $\mathfrak{N}_1(l_v)$ is rational by an argument like that used in Corollary 2.2. As $\dim \mathfrak{N}_1(l_v) - \dim \mathfrak{Z}_1(l_v) = 2(m-1)$, the induction is complete.

4.2. THEOREM. *Let n be a nilpotent Lie group. Then there is a rational field $\{\mathfrak{M}_l \colon l \in \mathfrak{N}^*\}$ of subalgebras such that for all l in a certain Zariski-open set $U \subseteq \mathfrak{N}^*$, $\mathfrak{M}_l$ is polarizing for l and $\mathfrak{H}_\infty(l) \subseteq \mathfrak{M}_l \subseteq \mathfrak{K}_\infty(l)$.*

PROOF. We let $E = \mathfrak{N}^*$ in Corollary 3.2 and find that $\mathfrak{H}^\infty(l)$, $\mathfrak{K}^\infty(l)$ depend rationally on l. Now the theorem is an immediate consequence of Lemma 4.1.

The Zariski-open set U in Theorem 4.2 is not $\mathrm{Ad}^*(N)$-invariant, and the $\mathfrak{M}_l$ are not covariant with repect to Ad. We may remedy these defects at the cost of a bit more work. It is implicit in the discussion from [6] cited previously (some further details are spelled out in [2]) that if the dimension of a generic $\mathrm{Ad}^*(N)$-orbit in $\mathfrak{N}$ is $\mathfrak{Z}$, then there is a subspace W of codimension $\mathfrak{Z}$ in $\mathfrak{N}$ and a birational map $F \colon W \times \mathbf{R}^d \to \mathfrak{N}$ such that

(1) $W \cap U = U_0$ is a nonvoid Zariski-open subset of W;

(2) for $l \in U_0$, $F(l \times \mathbf{R}^d) = \mathrm{Ad}^*(N)l$. Indeed, there is a birational map $G\colon W \times \mathbf{R}^d \to N$ such that for $l \in U_0$, $F(l, t) = \mathrm{Ad}^*(G(l, t))l$. Find rational $\mathfrak{M}_{l_0}$ for $l_0 \in U_0$ as in Theorem 4.2; for $l = F(l_0, t)$, define $\mathfrak{M}_l = \mathrm{Ad}(G(l, t))\mathfrak{M}_{l_0}$. Set $U_1 = F(U_0 \times \mathbf{R}^d)$. We have thus proved

4.3. THEOREM. *Let $\mathfrak{N}$ be a nilpotent Lie group. Then there are a Zariski-open subset $U_1 \subseteq \mathfrak{N}^*$ and a rational field $\{\mathfrak{M}_l \colon l \in \mathfrak{N}^*\}$ of subalgebras such that for all $l \in U_1$,*

(a) *$\mathfrak{M}_l$ is polarizing for l, and $\mathfrak{H}_\infty(l) \subseteq \mathfrak{M}_l \subseteq \mathfrak{K}_\infty(l)$,*

(b) *if $l' = (\mathrm{Ad}^* n)l$, $n \in N$, then $l' \in U_1$, and $\mathfrak{M}_{l'} = (\mathrm{Ad}\, n)\mathfrak{M}_l$.*

This theorem, and indeed all the results of this paper, hold equally for nilpotent algebraic groups over any field of characteristic 0.

4.4. EXAMPLE Let $\mathfrak{N}$ be the 4×4 upper triangular matrices with zeros on the diagonal, let E_{ij} have a 1 in the (i, j) place and zeros elsewhere, and define ideals

$$\mathfrak{A}_0 = (0), \quad \mathfrak{A}_1 = \mathbf{R}E_{14}, \quad \mathfrak{A}_2 = \mathfrak{A}_1 + \mathbf{R}E_{13}, \quad \mathfrak{A}_3 = \mathfrak{A}_2 + \mathbf{R}E_{12},$$
$$\mathfrak{A}_4 = \mathfrak{A}_3 + \mathbf{R}E_{24}, \quad \mathfrak{A}_5 = \mathfrak{A}_4 + \mathbf{R}E_{23}.$$

Then the $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3$ are abelian ideals and will lie in the $\mathfrak{M}_l$ described in [8]; for generic l, $\mathfrak{H}_\infty(l) = \mathfrak{K}_\infty(l) = \mathbf{R}E_{13} + \mathbf{R}E_{14} + \mathbf{R}E_{23} + \mathbf{R}E_{24}$.

BIBLIOGRAPHY

1. Ian Brown, *Dual topology of a nilpotent Lie group*, Ann. Sci. École Norm. Sup. (4^e Série) **6** (1973), 407–411.

2. L. Corwin and F. P. Greenleaf, *Fourier transforms of smooth functions on certain nilpotent groups*, J. Funct. Anal. **37** (1980), 203–217.

3. L. Corwin, F. P. Greenleaf and R. Penney, *A canonical formula for the distribution kernels of primary projections in L^2 of a nilmanifold*, Comm. Pure Appl. Math. **30** (1977), 355–372.

4. A. A. Kirillov, *Unitary representations of nilpotent Lie groups*, Uspehi Mat. Nauk **17** (1962), 57–110.

5. R. Penney, *Canonical objects in the Kirillov theory of nilpotent Lie groups*, Proc. Amer. Math. Soc. **66** (1977), 175–178.

6. L. Pukanszky, *Leçons sur les représentations des groups*, Dunod, Paris, 1967.

7. C. Rockland, *Hypoellipticity on the Heisenberg group-representation-theoretic criteria*, Trans. Amer. Math. Soc. **240** (1978), 1–52.

8. M. Vergne, *Construction de sous-algèbres subordonnées à un élément du dual d'une algèbre de Lie résoluble*, C. R. Acad. Sci. Paris Sér. A-B **270** (1970), A173–A175.

9. ______, *Construction de sous-algèbres subordonnées à un élément du dual d'une algèbre de Lie résoluble*, C. R. Acad. Sci. Paris Sér. A-B **270** (1970), A704–A707.

Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903

Courant Institute of Mathematical Sciences, New York University, New York, New York 10012