PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 1, January 1981

AN UPPER BOUND FOR THE SUM OF LARGE DIFFERENCES BETWEEN PRIME NUMBERS

R. J. COOK

ABSTRACT. Let p_n denote the nth prime number, $d_n = p_{n+1} - p_n$. We estimate the sum $\Sigma\, d_n$ taken over $p_n < x$, $d_n > x^\mu$ where $1/6 < \mu < 5/9$.

1. Introduction. There has been much work done on studying irregularities in the distribution of prime numbers. Let p_n denote the nth prime number and set $d_n = p_{n+1} - p_n$. Assuming the Riemann hypothesis Cramér [3] proved that $d_n = O(p_n^{1/2}\log p_n)$ and [4] that for any $\varepsilon > 0$

$$\sum_{p_n < x} d_n^2 \ll x \log^{3+\varepsilon} x, \tag{1}$$

where the constant implied by Vinogradov's notation depends only on ε, and Selberg [17] subsequently improved on this.

For $0 \leqslant \mu \leqslant 1$ let

$$S_\mu(x) = \sum_{p_n < x,\ d_n > x^\mu} d_n \tag{2}$$

and take $f(\mu)$ to be the least value such that for all $\varepsilon > 0$,

$$S_\mu(x) \ll x^{f(\mu)+\varepsilon} \quad \text{as } x \to \infty.$$

From (1) we see that the Riemann hypothesis implies

$$f(\mu)\begin{cases} = 0 & \text{for } \mu > 1/2, \\ \leqslant 1 - \mu & \text{for } 0 \leqslant \mu \leqslant 1/2. \end{cases}$$

Montgomery [15, pp. 130–132] has shown that the density hypothesis is sufficient to imply that $f(\mu) = 0$ for $\mu > 1/2$.

Since the Riemann hypothesis implies that $d_n = O(p_n^{1/2}\log p_n)$, Erdös asked whether $f(1/2) < 1$ and there has been much interest recently in estimating $f(1/2)$ (see [2], [11], [12], [16] and [20]). Heath-Brown [7] has given an unconditional proof that

$$\sum_{p_n < x} d_n^2 \ll x^{4/3}(\log x)^{10{,}000}$$

which gives

$$f(\mu)\begin{cases} = 0 & \text{for } \mu > 2/3, \\ \leqslant 4/3 - \mu & \text{for } 2/3 \geqslant \mu > 1/3, \end{cases} \tag{3}$$

Received by the editors March 13, 1978 and, in revised form, November 9, 1979.
AMS (MOS) subject classifications (1970). Primary 10H15.
Key words and phrases. Prime number, density theorem.

0002-9939/81/0000-0008/$03.00

while Huxley [**10**] has shown that $f(\mu) = 0$ for $\mu > 7/12$.

A recent result of Warlimont [**19**], when combined with Huxley's density theorem [**10**], shows that $f(\mu) < 1$ for $\mu > 1/6$. Warlimont's result depends on an inexplicit estimate of Halász and Turán [**6**] and so seems incapable of giving explicit upper bounds for $f(\mu)$ in the range $\mu > 1/6$. We use different estimates to improve the upper bound (3) throughout the interval $5/9 > \mu > 1/6$.

THEOREM. *We have* $f(\mu) \leqslant F(\mu)$ *where*

$$F(\mu) = \begin{cases} 19/18 - \mu/2 & \text{for } 47/99 < \mu \leqslant 5/9, \\ 10/7 - 9\mu/7 & \text{for } 3/8 < \mu \leqslant 47/99, \\ 1 - \mu/7 & \text{for } 7/32 < \mu \leqslant 3/8, \\ 11/10 - 3\mu/5 & \text{for } 1/6 < \mu \leqslant 7/32. \end{cases} \tag{4}$$

In particular, $f(1/2) \leqslant 29/36 = .80\dot{5}$ which is an improvement on Ivić's estimate [**13**], $f(1/2) \leqslant .809$.

I am grateful to Roger Heath-Brown, Martin Huxley and Aleksander Ivić for making manuscripts available prior to publication; I am indebted to the referee for many helpful suggestions and for pointing out a gap in my original argument.

2. Preliminaries. Let $\varepsilon > 0$ and $x > x_0(\varepsilon)$ be large. It is sufficient to obtain the corresponding estimates for a sum

$$\sum_{\substack{x < p_n \leqslant 2x \\ d_n > x^{\mu}}} d_n$$

since the theorem then follows on summing over intervals $[x, x/2], [x/2, x/4], \ldots$ and noticing that at $O(\log x)$ end-points we have $d_n = O(x^{7/12+\varepsilon})$: see Huxley [**10**].

The function

$$\psi(z) = \sum_{p^r \leqslant z} \log p$$

has an explicit formula (see Chandrasekharan [**1**, p. 120])

$$\psi(z) = z - \sum_{|\gamma| \leqslant T} z^{\rho}/\rho + E(z, T) \tag{5}$$

where the summation is taken over the nontrivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$, $0 < \alpha \leqslant 1$ and $E(z, T) = O(zT^{-1}\log^2 z)$ uniformly over the range $3 \leqslant T \leqslant z$.

We take

$$T = x^{\alpha}, \qquad U = 2x^{1-\mu}, \qquad \delta = \log(1 + 1/U) \ll U^{-1}, \tag{6}$$

and put

$$\Delta(y) = \psi\left(y + \frac{y}{U}\right) - \psi(y) - \frac{y}{U} + \sum_{(8)} \frac{(e^{\delta\rho} - 1)}{\rho} y^{\rho} - E\left(y + \frac{y}{U}\right) + E(y) \tag{7}$$

where $x < y \leqslant 2x$, the summation is over the zeros of $\zeta(s)$ in the region

$$|\gamma| \leqslant x^{\alpha}, \qquad \sigma_0 \leqslant \beta \leqslant 1, \tag{8}$$

and $\sigma_0 = \sigma_0(\mu)$ will be chosen later. From (5) we have

$$\Delta(y) = \sum_{(10)} \frac{(e^{\delta\rho} - 1)}{\rho} y^{\rho} \tag{9}$$

where the summation is over the zeros of $\zeta(s)$ in the region

$$|\gamma| \leqslant x^{\alpha}, \qquad 0 < \beta < \sigma_0. \tag{10}$$

Following Warlimont **[19]** we have

$$\int_x^{2x} |\Delta(y)|^2 \, dy \ll \delta^2 x \log^2 x \sum_{(10)} x^{2\beta} \ll U^{-2} x \log^2 x \sum_{(10)} x^{2\beta}. \tag{11}$$

The proof of the theorem depends on obtaining a lower bound for $\int |\Delta(y)|^2 \, dy$ in terms of large differences between primes, and an upper bound for the right side of (11) via zero-density estimates.

Let $N(\sigma, T)$ denote the number of zeros of $\zeta(s)$ in the rectangle $\sigma \leqslant \beta \leqslant 1$, $|\gamma| \leqslant T$ and write $N(T)$ for $N(0, T)$. Then, see Davenport **[5**, Chapter 15**]**, we have

$$N(T) \ll T \log T. \tag{12}$$

Since

$$\begin{aligned} \sum_{(10)} (x^{2\beta} - 1) &= 2 \sum_{(10)} \int_0^{\beta} x^{2\sigma} \log x \, d\sigma \\ &= 2 \int_0^{\sigma_0} x^{2\sigma} N(\sigma, x^{\alpha}) \log x \, d\sigma \end{aligned}$$

we have

$$\sum_{(10)} x^{2\beta} \ll x^{1+\alpha} \log x + \int_{1/2}^{\sigma_0} x^{2\sigma} N(\sigma, x^{\alpha}) \log x \, d\sigma. \tag{13}$$

3. Density theorems. These are upper bounds for $N(\sigma, T)$ which can be stated in either the form

$$N(\sigma, T) \ll T^{g(\sigma)(1-\sigma)} \log^A T \tag{14}$$

where A is an absolute constant, not necessarily the same at each occurrence, or in the form

$$N(\sigma, T) \ll T^{g(\sigma)(1-\sigma)+\eta} \tag{15}$$

where η is an arbitrary small positive constant. In (14) and (15) the implicit constants are independent of σ, although they may depend on A and η.

LEMMA 1. *For* $1/2 \leqslant \sigma \leqslant 1$, (14) *holds with* $g(\sigma) = 3/(2 - \sigma)$.

This is due to Ingham **[12]**.

LEMMA 2. *For* $3/4 \leqslant \sigma \leqslant 1$, (14) *holds with* $g(\sigma) = 3/(3\sigma - 1)$.

This is due to Huxley **[10]**.

LEMMA 3. *For* $3/4 \leqslant \sigma \leqslant 11/14$, (15) *holds with* $g(\sigma) = 1/(7\sigma - 5)$.

This is the case $k = 3$ of equation (1.8) of Jutila **[13]**.

LEMMA 4. *For* $11/14 \leqslant \sigma \leqslant 1$, (15) *holds with* $g(\sigma) = 9/(7\sigma - 1)$.

This is due to Heath-Brown [9].

LEMMA 5. *Let* $M(\psi, T) = \max|\zeta(s)|$ *where the maximum is taken over* $\operatorname{re} s \geqslant \psi$, $|\operatorname{im} s| \leqslant T$ *and* $|s - 1| \geqslant 1$. *Then for* $T \geqslant 2$

$$N(\sigma, T) \ll (BM(\psi, 8T)\log^5 T)^{2(1-\sigma)(3\sigma-1-2\psi)/(2\sigma-1-\psi)(\sigma-\psi)}\log^8 T \qquad (16)$$

provided that $1/2 \leqslant \psi \leqslant 1$ *and* $\sigma > (1 + \psi)/2$, *where B is a large absolute constant.*

This is Theorem 12.3 of Montgomery [15].

We combine Lemma 5 with estimates for $\zeta(s)$ obtained by van der Corput's method.

LEMMA 6. *If* $l \geqslant 3$, $L = 2^{l-1}$, $\sigma = 1 - l/(2L - 2)$ *then*

$$\zeta(\sigma + it) \ll t^{1/(2L-2)}\log t.$$

This is Theorem 5.14 of Titchmarsh [18].

LEMMA 7. *For* $17/18 \leqslant \sigma \leqslant 1$, (14) *holds with* $g(\sigma) = 1/(7\sigma - 6)$.

PROOF. Taking $l = 4$ in Lemma 6 we have

$$\zeta(5/7 + it) \ll t^{1/14}\log t.$$

This implies that the function

$$g(s) = \frac{\zeta(s)}{s^{1/14}\log s}\left(\frac{s-1}{s}\right)^2$$

which is holomorphic for $\operatorname{re} s > 1$, is uniformly bounded on the line $\operatorname{re} s = 5/7$. It is also uniformly bounded in the half-plane $\operatorname{re} s \geqslant 2$, so by the Phragmen-Lindelöf theorem it is uniformly bounded in $\operatorname{re} s \geqslant 5/7$. Hence

$$M(5/7, 8T) \ll T^{1/14}\log T.$$

Now apply Lemma 5 with $\psi = 5/7$. For $17/18 \leqslant \sigma \leqslant 1$ the exponent of B is bounded uniformly in σ so

$$g(\sigma) \leqslant \frac{2(3\sigma - 1 - 10/7)}{14(2\sigma - 1 - 5/7)(\sigma - 5/7)} = \frac{(21\sigma - 17)}{2(7\sigma - 6)(7\sigma - 5)} \leqslant \frac{1}{(7\sigma - 6)}.$$

Combining these lemmas, we have

$$N(\sigma, T) \ll T^{G(\sigma)(1-\sigma)+\eta} \qquad (1/2 \leqslant \sigma \leqslant 1) \qquad (17)$$

where

$$G(\sigma) = \begin{cases} 3/(2-\sigma), & 1/2 \leqslant \sigma \leqslant 3/4, \\ 3/(3\sigma - 1), & 3/4 \leqslant \sigma \leqslant 7/9, \\ 1/(7\sigma - 5), & 7/9 \leqslant \sigma \leqslant 11/14, \\ 9/(7\sigma - 1), & 11/14 \leqslant \sigma \leqslant 53/56, \\ 1/(7\sigma - 6), & 53/56 \leqslant \sigma \leqslant 1. \end{cases} \qquad (18)$$

Thus $G(\sigma)$ is a continuous function with $G(1/2) = 2$, increasing monotonically to the value $G(3/4) = 12/5$ and thereafter decreasing monotonically to the value $G(1) = 1$.

4. Estimation of an integral. For given μ satisfying $1/6 < \mu \leqslant 5/9$ we choose $\sigma^* = \sigma^*(\mu)$ to be the largest $\sigma^* < 1$ satisfying $(1-\mu)G(\sigma^*) = 1$. Thus

$$\sigma^* = \sigma^*(\mu) = \begin{cases} (7-\mu)/7, & 1/6 \leqslant \mu \leqslant 3/8, \\ (10-9\mu)/7, & 3/8 \leqslant \mu \leqslant 1/2, \\ (6-\mu)/7, & 1/2 \leqslant \mu \leqslant 5/9. \end{cases} \tag{19}$$

We take $\alpha = 1 - \mu + \varepsilon_1$ where $\varepsilon_1 > 0$ is a constant which will be chosen later. For a given small $\varepsilon_2 > 0$ we take $\sigma_0 = \sigma_0(\mu)$ to be the largest $\sigma_0 < 1$ satisfying

$$\alpha G(\sigma_0) = 1 - \varepsilon_2. \tag{20}$$

Since $G(\sigma)$ is a continuous decreasing function near $\sigma = 1$, for given $\delta > 0$ we have $0 < \sigma_0 - \sigma^* < \delta$ provided that ε_1 is chosen sufficiently small in terms of δ, and ε_2 sufficiently small in terms of ε_1.

LEMMA 8. *For any $\varepsilon > 0$ we can choose $\delta, \varepsilon_1, \varepsilon_2$ so that*

$$\int_{1/2}^{\sigma_0} x^{2\sigma} N(\sigma, x^\alpha) \log x \, d\sigma \ll x^{1+F(\mu)+\varepsilon} \quad \textit{as } x \to \infty.$$

PROOF. We break the range of integration up into subintervals $[1/2, 3/4]$, $[3/4, 7/9]$, $[7/9, 11/14]$, $[11/14, 53/56]$ and $[53/56, 1]$, stopping at σ_0. We use the upper bounds for $N(\sigma, x^\alpha)$ provided by (18), and note that the exponent is a continuous function of α and σ, and so uniformly continuous on the region $1/2 \leqslant \sigma \leqslant 1, 0 \leqslant \alpha \leqslant 1$.

Putting $a = 1 - \mu$ it is sufficient to prove that for any $\varepsilon > 0$

$$\int_{1/2}^{\sigma^*} x^{2\sigma} N(\sigma, x^a) \log x \, d\sigma \ll x^{1+F(\mu)+\varepsilon/2} \quad \textit{as } x \to \infty, \tag{21}$$

as the perturbations from σ^* to σ_0 and a to α can be absorbed in the exponent ε provided that ε_1, ε_2 and δ are sufficiently small as functions of ε. For we estimate the integrand in (21) as

$$\ll x^{2\sigma} x^{aG(\sigma)(1-\sigma)+\eta} \log x.$$

Between σ^* and σ_0 the function $G(\sigma)(1-\sigma)$ decreases so this is

$$\ll x^{2\sigma_0 + aG(\sigma^*)(1-\sigma^*)+\eta} \log x \ll x^{2\sigma_0 + (1-\sigma^*)+\eta} \log x \ll x^{1+\sigma^*+2\delta+\eta} \log x$$

and $\sigma^*(\mu) \leqslant F(\mu)$ for $1/6 \leqslant \mu \leqslant 5/9$. Replacing $a = 1 - \mu$ by $\alpha = 1 - \mu + \varepsilon_1$ in the estimates for the integrand will increase the exponents of x by a factor no larger than $1 + 3\varepsilon_1$, since $1 - \mu \geqslant 1/3$. Thus the lemma follows from the estimate (21).

The first interval contributes

$$\ll \max_{1/2 \leqslant \sigma \leqslant 3/4} x^{2\sigma + 3a(1-\sigma)/(2-\sigma)+\eta} \ll x^{3/2+3a/5+\eta},$$

since the exponent is an increasing function of σ.

The second interval contributes

$$\ll \max_{3/4 < \sigma < 7/9} x^{2\sigma + 3a(1-\sigma)/(3\sigma - 1) + \eta}.$$

The exponent is a convex function of σ so the maximum occurs at an end-point of the interval. Hence the contribution of the second interval is

$$\ll \max(x^{3/2+3a/5+\eta}, x^{14/9+a/2+\eta})$$

$$\ll \begin{cases} x^{3/2+3a/5+\eta}, & a > 5/9, \\ x^{14/9+a/2+\eta}, & a < 5/9. \end{cases}$$

The third interval contributes

$$\ll \max_{7/9 < \sigma < 11/14} x^{2\sigma + a(1-\sigma)/(7\sigma - 5) + \eta}$$

and again the exponent is a convex function of σ so this is

$$\ll \max(x^{14/9+a/2+\eta}, x^{11/7+3a/7+\eta}) \ll x^{14/9+a/2+\eta}$$

since $a = 1 - \mu > 2/9$.

For $4/9 < a < 1/2$, $\sigma^* \in [7/9, 11/14]$ and the total contribution of the interval $[1/2, \sigma^*]$ is $\ll x^{14/9+a/2+\eta}$.

For $1/2 < a < 5/8$, $\sigma^* \in [11/14, 53/56]$ and the contribution of the interval $[11/14, \sigma^*]$ is

$$\ll \max(x^{11/7+3a/7+\eta}, x^{2\sigma^* + 9a(1-\sigma^*)/(7\sigma^* - 1) + \eta})$$
$$= \max(x^{11/7+3a/7+\eta}, x^{1+\sigma^*+\eta}).$$

Now $\sigma^* = (1 + 9a)/7$ so the total contribution of the interval $[1/2, \sigma^*]$ is

$$\ll \begin{cases} x^{14/9+a/2+\eta}, & 1/2 < a < 52/99, \\ x^{1+\sigma^*+\eta}, & 52/99 < a < 5/8. \end{cases}$$

For $a > 5/8$ the contribution of the interval $[1/2, 53/56]$ is

$$\ll \begin{cases} x^{53/28+3a/35+\eta}, & a < 55/72, \\ x^{3/2+3a/5+\eta}, & a > 55/72. \end{cases}$$

Now $\sigma^* = (6 + a)/7$ and the contribution of the interval $[53/56, \sigma^*]$ to the integral is

$$\ll \max(x^{53/28+3a/35+\eta}, x^{1+\sigma^*+\eta}) \ll x^{1+\sigma^*+\eta} = x^{(13+a)/7+\eta}$$

since $a > 5/8$. Thus for $a > 5/8$ the integral $[1/2, \sigma^*]$ contributes

$$\ll \begin{cases} x^{3/2+3a/5+\eta}, & a > 25/32, \\ x^{(13+a)/7+\eta}, & a < 25/32, \end{cases}$$

and this completes the proof of the lemma.

From Lemma 8, (11) and (13) we have

$$\int_x^{2x} |\Delta(y)|^2 \, dy \ll x^{2\mu - 1 + \varepsilon}\{x^{1+\alpha} + x^{1+F(\mu)}\} \ll x^{2\mu + F(\mu) + \varepsilon} \tag{22}$$

provided that ε_1 is sufficiently small that $\alpha < F(\mu)$, as $F(\mu) > 1 - \mu$ for $1/6 < \mu < 5/9$.

5. Estimation of a sum. Suppose that $x < p_m < 2x$ and $p_{m+1} - p_m > x^{\mu}$. Then for

$$x < p_m < y < p_m + d_m/3 < p_{m+1} < 2x \tag{23}$$

we have $y + y/U < p_{m+1}$. Then there are no primes in the interval $(y, y + y/U)$ so

$$\psi(y + y/U) - \psi(y) = \sum_{y<n<y+y/U} \Lambda(n) \ll \log^2 x. \tag{24}$$

Further for any $\eta < \varepsilon_1$

$$E(y) - E(y + y/U) \ll x^{1-\alpha}\log x \ll x^{1-\eta}/U. \tag{25}$$

Observing that

$$|\exp(z) - 1| \leqslant e|z| \quad \text{for re } z \leqslant 1$$

we have

$$\sum_{(8)} \frac{(e^{\delta\rho} - 1)y^{\rho}}{\rho} \ll \delta \sum_{(8)} x^{\beta} \ll U^{-1} \int_{\sigma_0}^{1} x^{\sigma} N(\sigma, x^{\alpha}) \log x \, d\sigma. \tag{26}$$

For $\sigma \in [\sigma_0, 99/100]$ we use Lemmas 3, 4 and 7, as necessary, to estimate $N(\sigma, x^{\alpha})$. We have $\alpha G(\sigma) \leqslant 1 - \varepsilon_2$ throughout the interval and choosing $\eta = \varepsilon_2/300$,

$$\int_{\sigma_0}^{99/100} x^{\sigma} N(\sigma, x^{\alpha}) \log x \, d\sigma \ll x^{1-\varepsilon_2/100+\eta} \log x \ll x/\log x. \tag{27}$$

LEMMA 9. *For some positive constant d, $\zeta(s) \neq 0$ in the region*

$$\sigma = \text{re } s > 1 - d/(\log \tau)^{2/3}(\log\log \tau)^{1/3} \quad \textit{where } \tau = |\text{im } s| + 2.$$

This is Corollary 11.4 in Montgomery **[15]**.

We take, with some suitable constant D,

$$\sigma_1 = 1 - D(\log x)^{-3/4}. \tag{28}$$

Then $\zeta(s) \neq 0$ in the region $\sigma \geqslant \sigma_1$, $|t| \leqslant x^{\alpha}$ and so

$$\int_{99/100}^{1} x^{\sigma} N(\sigma, x^{\alpha}) \log x \, d\sigma = \int_{99/100}^{\sigma_1} x^{\sigma} N(\sigma, x^{\alpha}) \log x \, d\sigma.$$

For $\sigma \in [99/100, \sigma_1]$ we use Lemma 7 to estimate $N(\sigma, x^{\alpha})$ and note that $\alpha G(\sigma) \leqslant 1 - \varepsilon_2$ throughout the interval. Hence using (28),

$$\int_{99/100}^{\sigma_1} x^{\sigma} N(\sigma, x^{\alpha}) \log x \, d\sigma \ll x \log^{A+1} x \int_{99/100}^{\sigma_1} x^{-\varepsilon_2(1-\sigma)} \, d\sigma \ll x/\log x. \tag{29}$$

Combining (26), (27) and (29) we obtain

$$\sum_{(8)} \frac{(e^{\delta\rho} - 1)y^{\rho}}{\rho} \ll x/(U \log x). \tag{30}$$

Substituting this, (24) and (25) in (7) we obtain that for those y in the range (23),

$$|\Delta(y)| = |-y/U + O(x^{1-\eta}/U) + O(x/U \log x) + O(\log^2 x)| > x/2U.$$

Hence

$$\int_x^{2x} |\Delta(y)|^2 \, dy \gg \frac{x^2}{U^2} \sum_{\substack{x < p_m < 2x \\ d_m > x^\mu}} d_m,$$

and so, from (22),

$$\sum d_m \ll x^{-2} U^2 \int_x^{2x} |\Delta(y)|^2 \, dy \ll x^{F(\mu)+\varepsilon}.$$

References

1. K. Chandrasekharan, *Arithmetical functions*, Die Grundlehren der Math. Wissenschaften, Band 167, Springer-Verlag, New York and Berlin, 1970.

2. R. J. Cook, *On the occurrence of large gaps between prime numbers*, Glasgow Math. J. **20** (1979), 43–48.

3. H. Cramér, *Some theorems concerning prime numbers*, Ark. Mat. **15** (1920), 1–32.

4. ______, *On the order of magnitude of the difference between consecutive prime numbers*, Acta Arith. **2** (1937), 23–46.

5. H. Davenport, *Multiplicative number theory*, Markham, Chicago, Ill., 1967.

6. G. Halász and P. Turán, *On the distribution of the roots of the Riemann zeta and allied functions*. I, J. Number Theory **1** (1969), 121–137.

7. D. R. Heath-Brown, *The differences between consecutive primes*, J. London Math. Soc. **18** (1978), 7–13.

8. ______, *The differences between consecutive primes*. II, J. London Math. Soc. **19** (1979), 207–220.

9. ______, *Zero-density estimates for the Riemann zeta-function and Dirichlet L-functions*, J. London Math. Soc. **19** (1979), 221–232.

10. M. N. Huxley, *On the differences between consecutive primes*, Invent. Math. **15** (1972), 164–170.

11. ______, *A note on large gaps between prime numbers*, Acta Arith. **38** (1980), 63–68.

12. A. E. Ingham, *On the estimation of* $N(\sigma, T)$, Quart. J. Math. Oxford Ser. **11** (1940), 291–292.

13. A. Ivić, *On sums of large differences between consecutive primes*, Math. Ann. **241** (1979), 1–9.

14. M. Jutila, *Zero-density estimates for L-functions*, Acta Arith. **32** (1977), 55–62.

15. H. L. Montgomery, *Topics in multiplicative number theory*, Lecture Notes in Math., vol. 227, Springer-Verlag, Berlin, Heidelberg and New York, 1971.

16. C. J. Moreno, *The average size of gaps between primes*, Mathematika **21** (1974), 96–100.

17. A. Selberg, *On the normal density of primes in small intervals, and the difference between consecutive primes*, Arch. Math. Naturvid. **47** (1943), 87–105.

18. E. C. Titchmarsh, *The theory of the Riemann zeta-function*, Oxford Univ. Press, Oxford, 1951.

19. R. Warlimont, *Über die Häufigkeit grosser Differenzen konsecutiver Primzahlen*, Monatsh. Math. **83** (1977), 59–63.

20. D. Wolke, *Grosse Differenzen zwischen aufeinanderfolgenden Primzahlen*, Math. Ann. **218** (1975), 269–271.

Department of Pure Mathematics, University of Sheffield, Sheffield S10 2TN, England