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GENERALISED LIPSCHITZ CLASS OF FUNCTIONS AND THE ABSOLUTE SUMMABILITY OF FOURIER SERIES BY NÖRLUND MEANS

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ABSTRACT. The main aim of the paper is to investigate the relationship between certain generalised Lipschitz classes of functions and to discuss the absolute Nörlund summability of Fourier series of functions of the class L^q where $2 < q < \infty$.

1. The class $\operatorname{Lip}(\alpha, q)$ in $[a, b]$ is the class of functions $f(x)$ for which

$$\left\{\int_a^b |f(x+h) - f(x)|^q \, dx\right\}^{1/q} \leqslant C|h|^\alpha \qquad (q \geqslant 1, 0 < \alpha \leqslant 1)$$

where C is independent of h. $f(x)$ is said to belong to the class $\operatorname{Lip} \alpha$ $(0 < \alpha \leqslant 1)$ in $[a, b]$ if $f(x) - f(x-h) = O(h^\alpha)$ uniformly for $a \leqslant x - h < x \leqslant b$. Hardy and Littlewood [4] have shown that for $\alpha q > 1$ the class of functions $\operatorname{Lip}(\alpha, q)$ and $\operatorname{Lip}(\alpha - 1/q)$ are equivalent.

For a periodic function $f(t)$ with period 2π and integrable in the Lebesgue sense in $(-\pi, \pi)$, its Fourier series is

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=0}^{\infty} A_n(t),$$

where a_n and b_n are given by the usual Euler-Fourier formulae.

Let S_n denote the nth partial sum of the series Σa_n. The sequence to sequence transformation

$$t_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_{n-\nu} s_\nu \qquad (P_n > 0)$$

defines the sequence $\{t_n\}$ of Nörlund means of the sequence $\{s_n\}$ generated by the sequence of nonnegative numbers $\{p_n\}$ where $P_n = \Sigma_{\nu=0}^{n} p_\nu$ $(P_{-1} = p_{-1} = 0)$. The series Σa_n is said to be summable $|N, p_n|$ if $\Sigma_{n=1}^{\infty} |t_n - t_{n-1}| < \infty$. The particular case $p_0 = 1$ and $p_n = 0$ for $n > 0$ of the summability $|N, p_n|$ of Σa_n is its absolute convergence.

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We write

$$\phi(t) = f(x+t) + f(x-t) - 2f(x);$$

$$\alpha(t) = \sum_{\nu=0}^{\infty} p_\nu \cos \nu t; \qquad \beta(t) = \sum_{\nu=0}^{\infty} p_\nu \sin \nu t;$$

$$\alpha_n = \int_0^\pi \phi(t)\alpha(t)\cos nt\, dt; \qquad \beta_n = \int_0^\pi \phi(t)\beta(t)\sin nt\, dt;$$

$$\Delta\lambda_n = \lambda_n - \lambda_{n+1}; \qquad 1/q + 1/q' = 1.$$

C denotes a positive constant not necessarily the same at each occurrence.

N. Lusin [8] conjectured that if $f \in L^2$, then the Fourier series of f converges almost everywhere. This conjecture was based on his result that the conjugate function of f exists and is in L^2 for every f in L^2. Carleson [1] proved Lusin's conjecture and Hunt [5] extended the result for functions belonging to the class L^q where $q > 1$. One of the authors (see Lal [6]) obtained a result on the almost everywhere absolute Nörlund summability of Fourier series of functions belonging to L^q where $1 < q \leq 2$ which contains as a particular case a result on the absolute convergence of Fourier series for functions of the class L^q where $1 < q \leq 2$. In a recent paper, Okuyama has remarked that certain interesting results on the almost everywhere $|C, \alpha|$ summability of Fourier series can also be deduced from this theorem [10], [12]. In this paper we study the absolute Nörlund summability of Fourier series of functions belonging to the class L^q where $2 < q < \infty$ and establish the following:

THEOREM. *Let* $\{p_n\}$ *be a nonnegative and nonincreasing sequence such that* $\lim_{n\to\infty} p_n = 0$,

$$\sum_{k=n}^{\infty} |\Delta^2 p_k| = O(\Delta p_n), \tag{1.1}$$

and

$$\sum_{k=1}^{\infty} P_k^{2q/(q-2)} k^{-2} < \infty \qquad (2 < q < \infty). \tag{1.2}$$

Further let

$$\left\{ \int_{-\pi}^{\pi} |f(x+t) - f(x)|^q\, dx \right\}^{1/q} = O(\chi(|t|)), \tag{1.3}$$

and

$$|f(x+t) - f(x)|\, t^{1/q} = O(\chi(|t|)), \tag{1.4}$$

where for $u > 0$, $\chi(u)$ *is a positive function of* u *such that* $\chi(u)/u^{1/q}$ *increases as* u *increases,*

$$\sum_{k=n}^{\infty} \chi(k^{-1}) k^{-1/q'} = O(n^{1/q}\chi(n^{-1})), \tag{1.5}$$

and

$$\sum_{k=1}^{\infty} P_k^{-1} \chi(k^{-1}) k^{-1/2} < \infty. \tag{1.6}$$

Then the Fourier series $\Sigma A_n(t)$ *is summable* $|N, p_n|$.

Before proceeding to determine a set of sufficient conditions to be satisfied by $\chi(t)$ for which the condition (1.3) implies (1.4) (see the next section) we observe that the above theorem generalises earlier results due to Chow [2], [3] and McFadden [9].

2. Let $F(z) = F(re^{i\theta}) = \sum_{n=0}^{\infty} c_n z^n$ be regular for $r < 1$ and write

$$M_q(r, F) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|F(z)|^q\, d\theta\right)^{1/q}.$$

As $q \to \infty$, $M_q(r) \to M(r)$, the maximum modulus of $F(z)$.

We first note that if

$$\left\{\int_{-\pi}^{\pi}|f(x+t) - f(x)|^q\, dx\right\}^{1/q} = O(\chi(|t|)) \qquad (q > 1) \tag{2.1}$$

as $t \to 0$, where for $u > 0$, $\chi(u)$ is a positive monotonic increasing function of u such that

$$\int_1^{\pi/(1-r)} \frac{\chi((1-r)x)}{1+x^2}\, dx = O[\chi(1-r)], \tag{2.2}$$

and $\sum c_n e^{int}$ is the Fourier power series of a function $F(e^{it}) = f(t)$, then

$$M_q(r, F') = O[(1-r)^{-1}\chi(1-r)]. \tag{2.3}$$

We have

$$\begin{aligned} F'(re^{it}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{F(e^{iu})e^{iu}}{(e^{iu} - re^{it})^2}\, du \\ &= \frac{e^{-it}}{2\pi}\int_{-\pi}^{\pi}\frac{f(t+u)e^{iu}}{(e^{iu} - r)^2}\, du \\ &= \frac{e^{-it}}{2\pi}\int_{-\pi}^{\pi}\frac{e^{iu}}{(e^{iu} - r)^2}\{f(t+u) - f(t)\}\, du, \end{aligned}$$

and therefore

$$\begin{aligned} \left(\int_{-\pi}^{\pi}|F'(re^{it})|^q\, dt\right)^{1/q} &< \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{du}{|e^{iu} - r|^2}\left(\int_{-\pi}^{\pi}|f(t+u) - f(t)|^q\, dt\right)^{1/q} \\ &= O\left[\int_{-\pi}^{1-r}\frac{\chi(|u|)}{(1-r)^2 + u^2}\, du + \int_{1-r}^{\pi}\frac{\chi(u)}{(1-r)^2 + u^2}\, du\right] \\ &= O\left[(1-r)^{-1}\chi(1-r) + \frac{1}{1-r}\int_1^{\pi/(1-r)}\frac{\chi((1-r)x)}{1+x^2}\, dx\right] \\ &= O[(1-r)^{-1}\chi(1-r)], \end{aligned}$$

using (2.2). Thus (2.3) is established. We now show that if

$$M_q(r, F) = O[(1-r)^{-1}\chi(1-r)], \tag{2.4}$$

then

$$M(r) = O[(1-r)^{-1-1/q}\chi(1-r)]. \tag{2.5}$$

Let $\rho = \frac{1}{2}(1 + r)$. Then

$$|F(re^{i\theta})| = \left|\frac{\rho}{2\pi}\int_{-\pi}^{\pi}\frac{F(\rho e^{i\phi})e^{i\phi}}{\rho e^{i\phi} - re^{i\theta}}\,d\phi\right|$$

$$\leqslant \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|F(\rho e^{i\phi})|^{q}\,d\phi\right)^{1/q}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{d\phi}{|\rho e^{i\phi} - re^{i\theta}|^{q'}}\right)^{1/q'}$$

$$= O\left[(1-\rho)^{-1}\chi(1-\rho)\left\{\int_{0}^{\infty}\frac{dt}{[(\rho - r)^{2} + t^{2}]^{q'/2}}\right\}^{1/q'}\right]$$

$$= O[(1-\rho)^{-1}\chi(1-\rho)(\rho - r)^{-1/q}] = O[(1-r)^{-1-1/q}\chi(1-r)].$$

If in place of (2.4) we have (2.3), then it follows that

$$|F'(re^{i\theta})| = O[(1-r)^{-1-1/q}\chi(1-r)]. \tag{2.6}$$

Using (2.6), (1.5) and the conditions $t^{-1/q}\chi(t)$ increases as t increases and $t^{-1-1/q}\chi(t)$ decreases as t increases, we show that

$$|F(re^{i\theta+ih}) - F(re^{i\theta})| = O(h^{-1/q}\chi(h)), \tag{2.7}$$

provided that $0 < h \leqslant \frac{1}{2}$, $r \geqslant \frac{1}{2}$. Now

$$|F(re^{i\theta+ih}) - F(re^{i\theta})| = \left|\int_{re^{i\theta}}^{re^{i\theta+ih}}F'(z)\,dz\right| = |I_1 + I_2 + I_3|$$

where the path of the integral I_1 is the arc $(\theta, \theta + h)$ of the circle $|z| = r - h = \rho$, the path of I_2 is the straight line joining $re^{i\theta}$ to $(r - h)e^{i\theta}$ and the path of I_3 is the straight line joining $(r - h)e^{i(\theta+h)}$ to $re^{i(\theta+h)}$.

$$|I_1| \leqslant \int_{\theta}^{\theta+h}|F'(\rho e^{i\phi})|\,d\phi = \int_{0}^{h}|F'(\rho e^{i\theta+i\phi})|d\phi$$

$$= O[h(1-\rho)^{-1-1/q}\chi(1-\rho)] = O[h^{-1/q}\chi(h)],$$

since $t^{-1-1/q}\chi(t)$ decreases as t increases. And

$$|I_2| \leqslant \int_{\rho}^{\rho+h}|F'(te^{i\theta})|\,dt$$

$$= O\left[\int_{\rho}^{\rho+h}(1-t)^{-1-1/q}\chi(1-t)\,dt\right]$$

$$= O[h(1-\rho-h)^{-1-1/q}\chi(1-\rho-h)]$$

$$= O[h^{-1/q}\chi(h)],$$

if $1 - \rho - h > h$. If $1 - \rho - h < h$ using (1.5) and the fact that $t^{-1/q}\chi(t)$ increases as t increases, we have

$$|I_2| = O\left[\int_{\rho}^{1}(1-t)^{-1-1/q}\chi(1-t)\,dt\right]$$

$$= O\left[\int_{1/(1-\rho)}^{\infty}\frac{\chi(1/u)}{u^{1-1/q}}\,du\right]$$

$$= O[(1-\rho)^{-1/q}\chi(1-\rho)] = O(h^{-1/q}\chi(h)).$$

Similarly it can be shown that $|I_3| = O(h^{-1/q}\chi(h))$, and (2.7) follows.

3. The following lemmas are pertinent to the proof of our theorem.

LEMMA 1 [9]. *If $\{p_n\}$ is nonnegative and nonincreasing, then for $0 \leqslant a < b \leqslant \infty$, $0 < t \leqslant \pi$ and any n*

$$\left|\sum_{\nu=a}^{b} p_\nu e^{i(n-\nu)t}\right| \leqslant CP(t^{-1}); \tag{3.1}$$

$$p_n \sum_{\nu=0}^{n} \frac{P_\nu}{P_n} \leqslant CP(t^{-1}) \quad \text{for } t \leqslant \frac{1}{n}; \tag{3.2}$$

$$\sum_{\nu=n}^{\infty} \frac{\nu(p_\nu - p_{\nu+1})}{P_\nu P_{\nu-1}} \leqslant CP_{n-1}^{-1}; \tag{3.3}$$

and if we take $\gamma(t) = \sum_{\nu=0}^{\infty} p_\nu e^{i\nu t}$ then for t in (h, π)

$$|\gamma(t+2h) - \gamma(t)| \leqslant Cht^{-1}P(h^{-1}). \tag{3.4}$$

LEMMA 2 [9]. *If $\sum_{n=1}^{\infty} P_n^{2q/(q-2)} n^{-2} < C$ $(q > 2)$ then*

$$P_n^2 < Cn^{(q-2)/q} \tag{3.5}$$

and

$$\int_0^{1/n} P^2\left(\frac{1}{t}\right) dt < Cn^{-2/q}. \tag{3.6}$$

4. Proof of the theorem. Proceeding as in [7] we have

$$\begin{aligned}
\pi|t_n - t_{n-1}| \leqslant{} & \frac{|\alpha_n|}{P_{n-1}} + \frac{|\beta_n|}{P_{n-1}} + \frac{1}{P_{n-1}}\left|\int_0^{1/n} \phi(t)\left(\sum_{k=n}^{\infty} p_k \cos(n-k)t\right) dt\right| \\
& + \frac{p_n}{P_n P_{n-1}}\left|\int_0^{1/n} \phi(t)\left(\sum_{k=0}^{n-1} P_k \cos(n-k)t\right) dt\right| \\
& + \frac{1}{P_{n-1}}\left|\int_{1/n}^{\pi} \frac{\phi(t)}{2\sin(t/2)}\left(\sum_{k=n}^{\infty} (p_k - p_{k+1})\sin\left(n-k+\tfrac{1}{2}\right)t\right) dt\right| \\
& + \frac{p_n}{P_n P_{n-1}}\left|\int_{1/n}^{\pi} \frac{\phi(t)}{2\sin(t/2)}\left(\sum_{k=0}^{n-1} p_k \sin\left(n-k+\tfrac{1}{2}\right)t\right) dt\right| \\
& + \frac{p_n^2}{2P_{n-1}P_n}\left|\int_{1/n}^{\pi} \phi(t)\, dt\right| \\
={} & \sum_{i=1}^{n} I_i(n), \quad \text{say,}
\end{aligned}$$

and therefore to prove the theorem we have to show that

$$\sum_{n=2}^{\infty} I_i(n) < \infty \qquad (i = 1, 2, \ldots, 7). \tag{4.1}$$

Now

$$\begin{aligned}\sum_{n=2}^{\infty} I_1(n) &= \sum_{\nu=1}^{\infty} \sum_{n=2^{\nu-1}+1}^{2^\nu} |\alpha_n| P_{n-1}^{-1}\\ &< \sum_{\nu=1}^{\infty}\left(\sum_{n=2^{\nu-1}+1}^{2^\nu}\alpha_n^2\right)^{1/2}\left(\sum_{n=2^{\nu-1}+1}^{2^\nu} P_{n-1}^{-2}\right)^{1/2}\\ &< \sum_{\nu=1}^{\infty} 2^{\nu/2}P^{-1}(2^\nu)\left(\sum_{n=1}^{\infty}\alpha_n^2\sin^2\frac{n\pi}{2^{\nu+1}}\right)^{1/2}. \end{aligned} \tag{4.2}$$

Proceeding as in [7] and [11] and using (1.2), (1.3), (1.4), (3.1), (3.4) and (3.5) we have

$$\begin{aligned}\sum_{n=1}^{\infty}\alpha_n^2\sin^2 nh &< C\int_0^{\pi}|\phi(t+h)-\phi(t-h)|^2\alpha^2(t+h)\,dt\\ &\quad + C\int_{-h}^{h}\phi^2(t)\alpha^2(t+2h)\,dt + C\int_0^h\phi^2(t)\alpha^2(t)\,dt\\ &\quad + C\int_h^{\pi}\phi^2(t)|\alpha(t+2h)-\alpha(t)|^2\,dt\\ &< C\left\{\int_0^{\pi}|\phi(t+h)-\phi(t-h)|^q\,dt\right\}^{2/q}\left\{\int_0^{\pi}|\alpha(t+h)|^{2q/(q-2)}\,dt\right\}^{(q-2)/q}\\ &\quad + C\int_{-h}^{h}\chi^2(|t|)|t|^{-2/q}P^2((t+2h)^{-1})\,dt + C\int_0^h\chi^2(t)t^{-2/q}\alpha^2(t)\,dt\\ &\quad + Ch^2P^2(h^{-1})\int_h^{\pi}\chi^2(t)t^{-2-2/q}\,dt\\ &< C\chi^2(2h)\left\{\int_{(\pi+h)^{-1}}^{\infty}P^{2q/(q-2)}(t)t^{-2}\,dt\right\}^{(q-2)/q} + C\chi^2(h)P^2(h^{-1})h^{1-2/q}\\ &\quad + C\chi^2(h)h^{-2/q}\int_0^h P^2\left(\frac{1}{t}\right)dt + Ch^2P^2(h^{-1})\int_{1/\pi}^{1/h}\chi^2\left(\frac{1}{t}\right)t^{2/q}\,dt\\ &< C\chi^2(2h)\left[1+\sum_{k=1}^{\infty}P_k^{2q/(q-2)}k^{-2}\right]^{(q-2)/q}\\ &\quad + C\chi^2(h) + Ch^2P^2(h^{-1})\int_{1/\pi}^{1/h}\chi^2\left(\frac{1}{t}\right)t^{2/q}\,dt\\ &\leqslant C\chi^2(2h) + Ch^2P^2(h^{-1})\int_{1/\pi}^{1/h}\chi^2\left(\frac{1}{t}\right)t^{2/q}\,dt. \end{aligned}\tag{4.3}$$

From (4.2) and (4.3) we have

$$\begin{aligned}
\sum_{n=2}^{\infty} I_1(n) &< C\sum_{\nu=1}^{\infty} \frac{2^{\nu/2}\chi(\pi/2^\nu)}{P(2^\nu)} \\
&\quad + C\sum_{\nu=1}^{\infty} \frac{1}{2^{\nu/2}}\left(\int_{1/\pi}^{2^\nu} \chi^2\left(\frac{1}{t}\right)t^{2/q}\,dt\right)^{1/2} \\
&= C\sum_{n=1}^{\infty} \chi(n^{-1})P_n^{-1}n^{-1/2} + C\sum_{n=1}^{\infty}\frac{1}{n^{3/2}} \\
&\quad + C\sum_{\nu=1}^{\infty}\frac{1}{2^{\nu/2}}\sum_{m=1}^{\nu}\left(\sum_{n=2^{m-1}+1}^{2^m} n^{2/q}\chi^2\left(\frac{1}{n}\right)\right)^{1/2} \\
&< C + C\sum_{\nu=1}^{\infty}\frac{1}{2^{\nu/2}}\sum_{m=1}^{\nu}\chi\left(\frac{1}{2^{m-1}}\right)2^{(1/q+1/2)(m-1)} \\
&< C + C\sum_{m=1}^{\infty} 2^{(1/q+1/2)(m-1)}\chi\left(\frac{1}{2^{m-1}}\right)\sum_{\nu=m}^{\infty}\frac{1}{2^{\nu/2}} \\
&< C + C\sum_{m=1}^{\infty} 2^{(m-1)/q}\chi\left(\frac{1}{2^{m-1}}\right) \\
&< C + C\sum_{n=1}^{\infty}\chi(n^{-1})n^{-1/q'} < \infty. \qquad (4.4)
\end{aligned}$$

Similarly we can show that

$$\sum_{n=2}^{\infty} I_2(n) < \infty. \qquad (4.5)$$

Using the hypotheses (1.4), (1.5) of the theorem and the estimate (3.1) of Lemma 1 we have

$$\begin{aligned}
\sum_{n=2}^{\infty} I_3(n) &< C\sum_{n=2}^{\infty}\frac{1}{P_{n-1}}\int_0^{1/n}\chi(t)t^{-1/q}P(t^{-1})\,dt \\
&< C\sum_{n=2}^{\infty}\frac{1}{P_{n-1}}\sum_{k=n}^{\infty}\frac{P_k\chi(k^{-1})}{k^{2-1/q}} \\
&< C\sum_{n=1}^{\infty}\chi(n^{-1})n^{-1/q'} < \infty. \qquad (4.6)
\end{aligned}$$

Similarly by the hypothesis (1.4) and the estimate (3.2), we have

$$\sum_{n=2}^{\infty} I_4(n) < C\sum_{n=2}^{\infty}\frac{1}{P_{n-1}}\int_0^{1/n}\chi(t)t^{-1/q}P(t^{-1})\,dt < \infty, \qquad (4.7)$$

as in (4.6).

Using the hypotheses (1.1), (1.4) and (1.5) of the theorem we have

$$\begin{aligned}\sum_{n=2}^{\infty} I_5(n) &\leqslant C \sum_{n=2}^{\infty} \frac{1}{P_{n-1}} \int_{1/n}^{\pi} \frac{|\phi(t)|}{t^2} \left(\sum_{r=n}^{\infty} |\Delta^2 p_r| \right) dt \\ &\leqslant C \sum_{n=2}^{\infty} \frac{\Delta p_n}{P_{n-1}} \int_{1/n}^{\pi} \frac{\chi(t)}{t^{2+1/q}}\, dt \\ &\leqslant C + C \sum_{n=1}^{\infty} \frac{\Delta p_n}{P_{n-1}} \sum_{k=1}^{n} \int_{1/(k+1)}^{1/k} \frac{\chi(t)}{t^{2+1/q}}\, dt \\ &\leqslant C + C \sum_{k=1}^{\infty} \frac{1}{k} \int_{1/(k+1)}^{1/k} \frac{\chi(t)}{t^{2+1/q}}\, dt \\ &\leqslant C + C \sum_{k=1}^{\infty} \chi(k^{-1}) k^{-1/q'} < \infty. \end{aligned} \tag{4.8}$$

Again using the hypothesis (1.4) of the theorem and the estimate (3.1) of Lemma 1, we have

$$\begin{aligned}\sum_{n=2}^{\infty} I_6(n) &\leqslant C \sum_{n=2}^{\infty} \frac{p_n}{P_n P_{n-1}} \int_{1/n}^{\pi} \chi(t) t^{-1-1/q} P(t^{-1})\, dt \\ &\leqslant C + C \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{k=1}^{n} \frac{P_k \chi(k^{-1})}{k^{1-1/q}} \\ &\leqslant C + C \sum_{k=1}^{\infty} P_k \chi(k^{-1}) k^{-1/q'} \sum_{n=k}^{\infty} \frac{p_n}{P_n P_{n-1}} \\ &\leqslant C + C \sum_{k=1}^{\infty} \chi(k^{-1}) k^{-1/q'} < \infty. \end{aligned} \tag{4.9}$$

Finally

$$\sum_{n=2}^{\infty} I_7(n) \leqslant \sum_{n=1}^{\infty} \frac{p_n^2}{P_n P_{n-1}} < \infty. \tag{4.10}$$

Combining the estimates (4.4) through (4.10) the estimate in (4.1) follows.

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