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SPECTRAL MULTIPLICITY FOR TENSOR PRODUCTS OF NORMAL OPERATORS

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ABSTRACT. Two normal operators N_1 and N_2 are constructed such that for any pair m_1 and m_2 of their respective multiplicity functions, the 'convolution' $(m_1 * m_2)(\lambda) \equiv \Sigma\{m_1(\lambda_1) \cdot m_2(\lambda_2) | \lambda_1 \cdot \lambda_2 = \lambda\}$ fails to be a multiplicity function for the tensor product $N_1 \otimes N_2$.

Introduction. All operators discussed in this paper will be normal operators acting on separable Hilbert spaces. Let N be such an operator and let ν be one of its scalar spectral measures. The theory of spectral multiplicity associates with N a function $m: \mathbf{C} \to \mathbf{N} \cup \{\infty\}$ such that the equivalence class $[m]_\nu$ (i.e. the family of functions: $\mathbf{C} \to \mathbf{N} \cup \{\infty\}$ agreeing with m off a ν-null set) provides a complete unitary invariant for N. In particular, the spectrum of N and the measure class of ν are completely determined by $[m]_\nu$. On the other hand, it is only the equivalence class $[m]_\nu$ and not m, that is uniquely determined by N. It would be nice to choose 'canonical' representatives for these equivalence classes. For operators with countable spectra this is possible: we simply take m to be the unique representative of its equivalence class which is supported on the point spectrum of N.

If N_i $(i = 1, 2)$ have countable spectra and the canonical multiplicity functions m_i $(i = 1, 2)$ are chosen as in the preceding paragraph, then the function $m_1 * m_2$ defined by

$$(m_1 * m_2)(\lambda) \equiv \sum \{m_1(\lambda_1) \cdot m_2(\lambda_2) | \lambda_1 \cdot \lambda_2 = \lambda\} \tag{1}$$

provides a multiplicity function for the tensor product $N_1 \otimes N_2$. In their paper **[1]**, M. B. Abrahamse and T. L. Kriete constructed 'canonical' multiplicity functions for multiplication operators. They then asked whether formula (1) continues to provide a multiplicity function for the tensor product of multiplication operators with 'canonical' multiplicity functions m_1 and m_2 respectively. Example 2 of this note settles this question negatively. One might think that this could be rectified by changing the Abrahamse-Kriete multiplicity function. Example 3 of this note exhibits two multiplication operators N_1 and N_2 such that for every choice of multiplicity functions, equation (1) fails to provide a multiplicity function for $N_1 \otimes N_2$. This can be interpreted as meaning that there is, in general, no canonical way to choose representatives for multiplicity classes of normal operators.

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Notation and preliminaries. We recall the basic components of multiplicity theory for normal operators. For a fuller discussion, the reader should consult the introductory sections of **[1]** or **[2]**.

Let N be a normal operator with spectral measure E. A measure ν on $\mathbf{C}$ is said to be a *scalar spectral measure* for N in case $E(S) = 0$ if and only if $\nu(S) = 0$ for S a Borel set in $\mathbf{C}$. (All measures discussed in this note are completions of probability measures defined on complete separable metric spaces.) There exists a direct integral $\int_{\mathbf{C}}^{\oplus} H(\lambda)\, d\nu$ of Hilbert spaces having ν as a scalar spectral measure for N such that N is unitarily equivalent to the operator M on $\int_{\mathbf{C}}^{\oplus} H(\lambda)\, d\nu$ defined by $Mf(\lambda) = \lambda f(\lambda)$. The function $m\colon \mathbf{C} \to \mathbf{N} \cup \{\infty\}$ defined by $m(\lambda) = \dim H(\lambda)$ is called a *multiplicity function* for N; its equivalence class $[m]_\nu$ is called the *multiplicity class* of N and constitutes a complete unitary invariant for N.

Let (X, μ) be a measure space. As usual, we write $L^\infty(\mu)$ ($L^2(\mu)$) for the collection of equivalence classes of bounded measurable (respectively square-integrable) complex-valued functions on X. Let $\phi \in L^\infty(\mu)$. Then the multiplication operator M_ϕ is defined by $M_\phi f = \phi f$ ($f \in L^2(\mu)$). It is a bounded normal operator having $\nu \equiv \mu \circ \phi^{-1}$ as a scalar spectral measure. The papers **[1]** and **[2]** address the problem of constructing multiplicity functions for such operators. We close this section by summarizing these results.

If φ is a Borel representative of ϕ, we define $m_\varphi\colon \mathbf{C} \to \mathbf{N} \cup \{\infty\}$ by taking $m_\varphi(\lambda)$ to be the cardinality of the set $\varphi^{-1}(\lambda)$. (All infinite cardinalities are identified.)

PROPOSITION 1. *Let $\phi \in L^\infty(X, \mu)$, take M_ϕ to be the corresponding multiplication operator, and set $\nu = \mu \circ \phi^{-1}$.*

(1) *There is a Borel representative φ_0 of ϕ such that m_{φ_0} is a multiplicity function for M_ϕ.*

(2) *If φ is any Borel representative of ϕ, then m_φ is ν-measurable and $m_\varphi(\lambda) \geq m_{\varphi_0}(\lambda)$ for ν-almost all $\lambda \in \mathbf{C}$.*

(3) *Given a Borel multiplicity function m for M_ϕ, there is a Borel representative φ of ϕ such that $m_\varphi(\lambda) \leq m(\lambda)$ for all $\lambda \neq 0$.*

(4) *In order that M_ϕ have uniform multiplicity one, it is necessary and sufficient that some (hence every) representative φ of ϕ be one-to-one on a set of full μ-measure.*

PROOF. Statements (1) and (2) constitute Theorem 4.1 of **[2]**; statement (4) is contained in Corollary 4.1 of that paper. To prove (3), choose φ_0 as in (1), and set $E = \{\lambda \in \mathbf{C} \mid m_{\varphi_0}(\lambda) > m(\lambda)\}$. Since $\nu(E) = 0$, there is a Borel set F containing E with $\nu(F) = 0$. Set

$$\varphi(x) = \begin{cases} \varphi_0(x) & \text{if } x \notin \varphi_0^{-1}(F), \\ 0 & \text{if } x \in \varphi_0^{-1}(F). \end{cases}$$

For $\lambda \neq 0$, we have $\varphi^{-1}(\lambda) \subseteq \varphi_0^{-1}(\lambda)$ so $m_\varphi(\lambda) \leq m_{\varphi_0}(\lambda)$. Moreover, if $\lambda \in E \sim \{0\}$, $\varphi^{-1}(\lambda) = \varnothing$ so $m_\varphi(\lambda) = 0$. This completes the proof. □

REMARK. If ν is not totally atomic the Borel representative φ of (3) can be chosen to satisfy $m_\varphi \equiv m$.

Finally we recall the construction of the Abrahamse-Kriete multiplicity function [1] for M_ϕ. Given $\lambda \in$ ess ran ϕ, a point $x \in X$ is said to belong to the *essential preimage*, $\phi_{\text{ess}}^{-1}(\lambda)$, *of* ϕ *at* λ in case

$$\varliminf_{\delta \to 0} \frac{\mu(V \cap \phi^{-1}(B_\delta(\lambda)))}{\mu(\phi^{-1}(B_\delta(\lambda)))}$$

is strictly positive for every neighborhood V of x; here $B_\delta(\lambda)$ denotes the closed ball of radius δ about λ. The function $\overline{m}_\phi$: $\mathbf{C} \to \mathbf{N} \cup \{\infty\}$ (or simply $\overline{m}$ if ϕ is understood) is defined as

$$\overline{m}(\lambda) = \begin{cases} \text{cardinality of } \phi_{\text{ess}}^{-1}(\lambda) & \text{if } \lambda \in \text{ess ran } \phi, \\ 0 & \text{if } \lambda \notin \text{ess ran } \phi. \end{cases}$$

It is shown in [1] that $\overline{m}$ is a multiplicity function for M_ϕ; in the sequel, it will be referred to as the *Abrahamse-Kriete multiplicity function* for M_ϕ.

Noncanonicity of multiplicity functions. Given functions m_i: $\mathbf{C} \to \mathbf{N} \cup \{\infty\}$ $(i = 1, 2)$, we adopt equation (1) as the definition of $m_1 * m_2$.

PROPOSITION 2. *Let* N_1 *and* N_2 *be normal operators having multiplicity functions* m_1 *and* m_2 *respectively. Let* ν *be a scalar spectral measure for* $N_1 \otimes N_2$, *and suppose* m *is one of its multiplicity functions. Then* $(m_1 * m_2)(\lambda) \geqslant m(\lambda)$ *for* ν*-almost all* λ.

PROOF. Let ν_i be a scalar spectral measure for N_i; up to absolute continuity, the measure ν is given by $\nu(E) = (\nu_1 \times \nu_2)\{(\lambda_1, \lambda_2) \in \mathbf{C} \times \mathbf{C} | \lambda_1 \cdot \lambda_2 \in E\}$. By lowering the values of m_i on a ν_i-null set, we can make m_1 and m_2 Borel measurable. Since this can only decrease the value of $(m_1 * m_2)(\lambda)$, we may as well assume m_1 and m_2 are Borel to start with. Since every normal operator is unitarily equivalent to a multiplication operator, we may take N_i to be a multiplication operator M_{ϕ_i} acting on a Hilbert space $L^2(X_i, \mu_i)$. Apply Proposition 1(3) to construct a Borel representative φ_i of ϕ_i such that $m_{\varphi_i}(\lambda) \leqslant m_i(\lambda)$ for $\lambda \neq 0$, let φ: $X_1 \times X_2 \to \mathbf{C}$ by $\varphi(x_1, x_2) = \varphi_1(x_1) \cdot \varphi_2(x_2)$ and write $\phi = [\varphi]_{\mu_1 \times \mu_2}$. Then $N_1 \otimes N_2$ is unitarily equivalent to M_ϕ acting on $L^2(X_1 \times X_2, \mu_1 \times \mu_2)$ and so $m(\lambda) \leqslant m_\varphi(\lambda)$ for ν-almost all λ by Proposition 1(2). But for $\lambda \neq 0$, we have $m_\varphi(\lambda) = (m_{\varphi_1} * m_{\varphi_2})(\lambda) \leqslant (m_1 * m_2)(\lambda)$ so the proof is complete as long as $\{0\}$ is not an atom for ν. On the other hand, if $\{0\}$ is an atom for ν, then $m(0) = \dim \ker(N_1 \otimes N_2)$ which is easily seen to be less than or equal to $m_1 * m_2(0)$. □

Proposition 2 means that $m_1 * m_2$ can only fail to be a multiplicity function for $N_1 \times N_2$ by being too large, i.e. by having the strict inequality $(m_1 * m_2)(\lambda) > m(\lambda)$ hold on a set of λ which does not have ν measure zero. The three examples which follow illustrate successively more pathological instances of this behavior.

In each of these examples, μ_i $(i = 1, 2)$ is a measure supported on [0, 1], $\varphi_1 = \varphi_2$ is the identity function on [0, 1], $\phi_i = [\varphi_i]_{\mu_i}$, and N_i is the multiplication operator M_{ϕ_i} acting on $L^2([0, 1], \mu_i)$. Thus $N_1 \otimes N_2$ is unitarily equivalent to the multiplication operator $M_{\phi_1 \otimes \phi_2}$ acting on $L^2([0, 1] \times [0, 1], \mu_1 \times \mu_2)$. It will be convenient to let p: $\mathbf{R}^2 \to \mathbf{R}$ denote multiplication. Of course, μ_i is a scalar spectral measure for N_i and $\nu \equiv (\mu_1 \times \mu_2) \circ p^{-1}$ provides a scalar spectral measure for $N_1 \otimes N_2$.

EXAMPLE 1. Let $\mu_1 = \mu_2$ be the point-mass measure based at 1, and let $m_1 = m_2$: $\mathbf{C} \to \mathbf{N}$ be the constant function 1. Then m_i is a multiplicity function for N_i, but $m_1 * m_2 \equiv \infty$ is not a multiplicity function for $N_1 \otimes N_2 = I$. On the other hand, $\overline{m}_1 = \overline{m}_2$ equals the characteristic function of the singelton set $\{1\}$, so $\overline{m}_1 * \overline{m}_2$ is a multiplicity function for $N_1 \otimes N_2$.

EXAMPLE 2. Let $A_1 = \{e^{-q} | q > 0, q \in \mathbf{Q}\}$ and $A_2 = \{e^{-q\sqrt{2}} | q > 0, q \in \mathbf{Q}\}$. Let μ_i be a totally atomic measure on [0, 1] having A_i as its set of atoms. Since A_i is dense in [0, 1], the Abrahamse-Kriete multiplicity function $\overline{m}_i$ turns out to be the characteristic function $X_{[0,1]}$. Thus $(\overline{m}_1 * \overline{m}_2)(\lambda) = \infty$ for each $\lambda \in (0, 1)$. On the other hand, $\mu_1 \times \mu_2$ is supported on $A_1 \times A_2$ and p is one-to-one on $A_1 \times A_2$, so $N_1 \otimes N_2$ has uniform multiplicity one. Thus $\overline{m}_1 * \overline{m}_2$ is not a multiplicity function for $N_1 \otimes N_2$, and we have a counterexample to the conjecture in §7 of [1]. Note however that $m_i \equiv X_{A_i}$ provides a multiplicity for N_i such that $m_1 * m_2$ is a multiplicity function for $N_1 \otimes N_2$.

EXAMPLE 3. We construct measures μ_i $(i = 1, 2)$ such that for any choice m_i of multiplicity function for N_i, $m_1 * m_2$ fails to be a multiplicity function for $N_1 \otimes N_2$. We need the following lemma.

LEMMA. *There exist four uncountable, closed subsets A, B, C, D of* [0, 1] *with the following properties*:

(1) *p is one-to-one on $(A \times D) \cup (B \times C) \cup (B \times D)$.*

(2) *For each $b \in B$, there is a unique $d \in D$ such that $b \cdot d \in A \cdot C$ and for each $d \in D$, there is a unique $b \in B$, such that $b \cdot d \in A \cdot C$.*

(3) $A \cdot C \subseteq B \cdot D$.

PROOF. For a member $x \in [0, 10]$, we let x_n $(n = 0, 1, 2, \dots)$ denote the nth digit in its decimal expansion. (Expansions ending in nines will not be considered.) Let A_0, B_0, C_0, D_0 be the sets of numbers having the decimal expansions indicated in the following table.

	x_0	x_{2n-1} $(n > 0)$	x_{2n} $(n > 0)$
A_0	1	0 or 1	1
B_0	0	0	0, 1, 3 or 4
C_0	3	0 or 3	3
D_0	4	0, 1, 3 or 4	$4 - x_{2n-1}$

It is easy to check that A_0, B_0, C_0, D_0 satisfy conditions analogous to (1), (2) and (3) of the lemma with addition in place of multiplication. Set $A = \{e^{-x} | x \in A_0\}$, $B = e^{-B_0}$, $C = e^{-C_0}$, $D = e^{-D_0}$. □

By Theorem 8.1 of [3], we know that the sets A and C can be equipped with nonatomic measures α and γ respectively. Define a measure β on B by setting $\beta(S) = (\alpha \times \gamma)(p^{-1}(S \cdot D))$. Similarly let δ be defined on D by $\delta(S) = (\alpha \times \gamma)(p^{-1}(B \cdot S))$. Finally we take $\mu_1 = (\alpha + \beta)/2$, $\mu_2 = (\gamma + \delta)/2$.

We first show that $N_1 \otimes N_2$ has uniform multiplicity 1. To see this, let $E = B \times D \cap p^{-1}(A \cdot C)$. By (2) of the lemma, each section $E_d \equiv \{b \in B | (b, d) \in E\}$ of E in a singleton, and hence has μ_1 measure zero. It follows by Fubini's theorem that $\mu_1 \times \mu_2(E) = 0$. By (1) and (3) of the lemma, p is one-to-one on $(A \cup B) \times (C \cup D) \sim E$. Since $\mu_1 \times \mu_2$ is supported on this set, we conclude by Proposition 1(4) that $N_1 \otimes N_2$ has uniform multiplicity one.

Suppose next that m_1 and m_2 are multiplicity functions for N_1 and N_2 respectively. Since μ_i is a scalar spectral measure for N_i, there is a set X_i of full measure in [0, 1] on which m_i is identically one. If λ belongs to both of the sets $(X_1 \cap A) \cdot (X_2 \cap C)$ and $(X_1 \cap B) \cdot (X_2 \cap D) \cap A \cdot C$, then $(m_1 * m_2)(\lambda) \geqslant 2$. We will show that these sets have full ν-measure in $A \cdot C$. Now

$$\nu(A \cdot C) = (\mu_1 \times \mu_2)(p^{-1}(A \cdot C)) = (\mu_1 \times \mu_2)(A \times C \cup E) = \tfrac{1}{4}.$$

Thus we will know that $(m_1 * m_2)(\lambda) > 1$ on a set of positive ν-measure, so that in view of the preceding paragraph, $m_1 * m_2$ cannot be a multiplicity function for $N_1 \otimes N_2$.

It remains to show that

$$\nu\big[(X_1 \cap A) \cdot (X_2 \cap C)\big] = \nu\big[(X_1 \cap B) \cdot (X_2 \cap D) \cap A \cdot C\big] = \tfrac{1}{4}.$$

The first set is easy: $\nu((X_1 \cap A) \cdot (X_2 \cap C)) \geqslant (\mu_1 \times \mu_2)[(X_1 \cap A) \times (X_2 \cap C)] = \mu_1(A) \cdot \mu_2(C) = \frac{1}{4}$. To handle the second set, we first note that if $S \subseteq A \times C$, then $(\mu_1 \times \mu_2)(S) = \frac{1}{4}(\alpha \times \gamma)(S)$. It follows that $\nu[(B \sim X_1) \cdot D \cap A \cdot C] = \frac{1}{4}(\alpha \times \gamma)(p^{-1}[(B \sim X_1) \cdot D])$ which is by definition of β equal to $\frac{1}{4}\beta(B \sim X_1)$. Since X_1 has full measure, we conclude that $\nu[(B \sim X_1) \cdot D \cap A \cdot C] = 0$. A similar argument shows that $\nu[B \cdot (D \sim X_2) \cap A \cdot C] = 0$. Thus

$$\nu\big[(X_1 \cap B) \cdot (X_2 \cap D) \cap A \cdot C\big] = \nu(B \cdot D \cap A \cdot C) = \nu(A \cdot C) = \tfrac{1}{4}$$

as desired.

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