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UNIFORM APPROXIMATION BY RATIONAL MODULES ON NOWHERE DENSE SETS

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ABSTRACT. We prove that the rational module $\mathfrak{R}(X)\mathfrak{P}_1$ is always uniformly dense in $C(X)$ if the compact set X has no interior.

Let X be a compact subset of the complex plane **C**. Let the module $\mathfrak{R}(X)\mathfrak{P}_m$ be the space

$$\mathfrak{R}(X) + \mathfrak{R}(X)\bar{z} + \cdots + \mathfrak{R}(X)\bar{z}^m = \{r_0(z) + r_1(z)\bar{z} + \cdots + r_m(z)\bar{z}^m\},$$

where each r_i is a rational function with poles off X. These modules arise naturally in the study of rational approximation in Lipschitz norms and some other norms. They have been treated by O'Farrell [2] and Wang [3], [4]. A basic question was asked in [2] and [3]: Is $\mathfrak{R}(X)\mathfrak{P}_1$ dense in $C(X)$ for all X without interior? The purpose of this note is to show that the answer is yes.

We use the symbol $\bar{\partial}$ for the first order partial differential operator $\partial/\partial\bar{z}$ and let $\bar{\partial}^2 = \bar{\partial} \circ \bar{\partial}$. Lebesgue two-dimensional measure in **C** will be denoted by m.

Define $K(z, \omega) = (\bar{z} - \bar{\omega})/(z - \omega)$ for $z \neq \omega$, $K(z, z) = 0$. Then K is a Borel function on $\mathbf{C} \times \mathbf{C}$, $|K| \leqslant 1$ everywhere. If μ is a finite, compactly supported measure on **C**, we define

$$\check{\mu}(\omega) = \int K(z, \omega)\, d\mu(z).$$

Clearly $\check{\mu} \in L^\infty(m)$. If $\hat{\mu}$ is the Cauchy transform of μ, then $\check{\mu}(\omega) = \widehat{\bar{z}\mu}(\omega) - \bar{\omega}\hat{\mu}(\omega)$.

The following lemma supplies the main tool in studying approximation by rational modules.

LEMMA 1. *Let $f \in C_c^{(2)}$. Then for all $\omega \in \mathbf{C}$,*

$$f(\omega) = \frac{1}{\pi}\int_{\mathbf{C}} \bar{\partial}^2 f(z) \cdot \frac{\bar{z} - \bar{\omega}}{z - \omega}\, dm(z).$$

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PROOF.

$$
\begin{aligned}
\int_{\mathbf{C}} \bar\partial^2 f(z) \cdot \frac{\bar z - \bar\omega}{z - \omega}\, dm(z) - \pi f(\omega) \\
&= \int_{\mathbf{C}} \bar\partial^2 f(z) \cdot \frac{\bar z - \bar\omega}{z - \omega}\, dm(z) + \int_{\mathbf{C}} \bar\partial f(z) \cdot \frac{1}{z - \omega}\, dm(z) \\
&= \int_{\mathbf{C}} \bar\partial\left[\bar\partial f(z) \cdot (\bar z - \bar\omega)\right] \cdot \frac{1}{z - \omega}\, dm(z) \\
&= -\pi \bar\partial f(\omega) \cdot (\bar\omega - \bar\omega) = 0.
\end{aligned}
$$

LEMMA 2. *Let μ be a measure on* $\mathbf{C}$ *with compact support. Let $f \in C_c^{(2)}$. Then*

$$\int f\, d\mu = \frac{1}{\pi} \int \bar\partial^2 f \cdot \check\mu\, dm.$$

PROOF. Use Lemma 1 and Fubini's theorem.

COROLLARY. *If μ is a measure on* $\mathbf{C}$ *with compact support such that $\check\mu = 0$ a.e. (m), then $\mu = 0$.*

LEMMA 3. *Let μ be a measure on* $\mathbf{C}$ *with compact support. Then $\check\mu$ is continuous at every point $\omega \in \mathbf{C}$ such that $\mu(\{\omega\}) = 0$; in particular, $\check\mu$ is continuous except perhaps on a countable set.*

PROOF. Let $\omega_n \to \omega$. Then $K(z, \omega_n) \to K(z, \omega)$ for all $z \neq \omega$. Thus $K(\cdot, \omega_n) \to K(\cdot, \omega)$ a.e. (μ) and boundedly, provided $\mu(\{\omega\}) = 0$, so $\check\mu(\omega_n) \to \check\mu(\omega)$.

LEMMA 4. *Let μ be a measure on X. Then $\mu \perp \mathfrak{R}(X)\mathfrak{p}_1$ if and only if $\check\mu = 0$ off X.*

PROOF. If $\omega \notin X$, $z \to (\bar z - \bar\omega)/(z - \omega)$ is a function in $\mathfrak{R}(X)\mathfrak{p}_1$, so $\int K(z, \omega)\, d\mu(z) = 0$ whenever $\mu \perp \mathfrak{R}(X)\mathfrak{p}_1$.

If $\check\mu = 0$ off X, then for any $f = g + h\bar z$, where g and h are holomorphic in a neighborhood U of X, we can choose $\phi \in C_c^{(2)}(U)$ such that $\phi = 1$ in a neighborhood of X. Then

$$
\begin{aligned}
\int f\, d\mu = \int f\phi\, d\mu = \frac{1}{\pi} \int \bar\partial^2 (f\phi) \cdot \check\mu\, dm = \frac{1}{\pi} \int_X \bar\partial^2 (f\phi) \cdot \check\mu\, dm \\
= \frac{1}{\pi} \int_X \bar\partial^2 f \cdot \check\mu\, dm = 0,
\end{aligned}
$$

since $\bar\partial^2 f = 0$ on X.

THEOREM. *Let X be a compact set with no interior. Then $\mathfrak{R}(X)\mathfrak{p}_1$ is dense in $C(X)$.*

PROOF. If $\mu \perp \mathfrak{R}(X)\mathfrak{p}_1$, then $\check\mu = 0$ in $\mathbf{C} \setminus X$ by Lemma 4, hence $\check\mu = 0$ on $X \setminus S$ where S is at most countable, by Lemma 3, so $\mu = 0$ by the corollary to Lemma 2.

Davie's theorem in **[1]** asserts that for any compact set Y with boundary X, $C(X)$ is the closed linear span of $\mathfrak{R}(X)$ and $A(Y)$, the algebra of all continuous functions on Y which are analytic on the interior of Y. We may strengthen this result as in the following corollary. We put $\mathfrak{R}(X)\hat{} = \{(f_{|X})\hat{}: f \in \mathfrak{R}(X)\}$. Obviously $\mathfrak{R}(X)\hat{} \subset A(Y)$.

COROLLARY. *Let X be a compact set with no interior. Then $C(X)$ is the closed linear span of $\mathcal{R}(X)$ and $\mathcal{R}(X)\hat{}$.*

PROOF.

$$\mu \perp \mathcal{R}(X)\bar{\mathfrak{p}}_1 \Leftrightarrow \mu \perp \mathcal{R}(X) \text{ and } \hat{\mu} \perp \mathcal{R}(X) \qquad (\text{see } [\mathbf{2}] \text{ or } [\mathbf{3}])$$
$$\Leftrightarrow \mu \perp \mathcal{R}(X) + \mathcal{R}(X)\hat{}.$$

The method in this note can be applied to other types of rational modules. The result will appear elsewhere.

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