ON COMPACT MULTIPLIERS OF BANACH ALGEBRAS

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Abstract. We show that if the maximal ideal space of a commutative semisimple Banach algebra $B$ contains no isolated points, then every compact multiplier is trivial.

In [1] and [2] it was shown that if a commutative semisimple Banach algebra $B$ satisfies certain regularity conditions and if the maximal ideal space of $B$ contains no isolated points, then every compact multiplier of $B$ is trivial. ($T$ is a multiplier of $B$ if $T$ is a linear operator satisfying $T(fg) = f \cdot Tg$ for $f, g \in B$.) In this note we show that the regularity conditions used in [1] and [2] are unnecessary. Specifically we prove the following.

Theorem. Let $B$ be a commutative semisimple Banach algebra and $T$ a compact multiplier of $B$. If the maximal ideal space of $B$ contains no isolated points, then $T = 0$.

Proof. Let $X$ denote the maximal ideal space of $B$ and assume that $X$ contains no isolated points. Since $T$ is a multiplier of $B$, there exists a complex-valued continuous function $u$ on $X$ with $(Tf)'(x) = u(x)f(x)$ for all $f \in B$ and $x \in X$. We will show first that for each $x \in X$, $u(x)$ is an eigenvalue of the adjoint $T^*$ of $T$. Indeed, for $x \in X$, let $e_x$ denote the linear functional in $B^*$ which is evaluation at $x$, i.e. $e_x(f) = f(x)$. Then for each $f \in B$, we have $(T^*e_x)(f) = e_x(Tf) = (Tf)'(x) = u(x)f(x) = u(x)e_x(f)$. Thus $T^*e_x = u(x)e_x$ which proves that $u(x)$ is an eigenvalue of $T^*$.

Now $T$, and hence $T^*$, is compact, so that the spectrum of $T^*$, $\sigma(T^*)$, is a denumerable set with 0 as its only possible limit point. $\sigma(T^*)$ also has the property that every nonzero element in $\sigma(T^*)$ is an eigenvalue of finite multiplicity. Suppose $x_0$ is a point in $X$ which is not an isolated point. We claim that $u(x_0) = 0$. Indeed, suppose $u(x_0) \neq 0$. Since $u$ is a continuous function on $X$ and $x_0$ is a limit point of $X$, for each positive integer $n$, there exists an element $x_n$, $x_0 \neq x_n \in X$, with $|u(x_n) - u(x_0)| < 1/n$. However, each nonzero eigenvalue of $T^*$ has finite multiplicity and so $u(x_n) = u(x_0)$ for only finitely many $n$. Therefore $u(x_0)$ is a limit point of $\{u(x_n)\} \subset \sigma(T^*)$. However 0 is the only possible limit point of $\sigma(T^*)$ since $T^*$ is compact. This contradiction shows that $u(x_0) = 0$. Since no element in $X$ is an isolated point, by hypothesis, we conclude that $u(x) = 0$ for all $x \in X$. Hence
\[(Tf)^\ast(x) = u(x)\hat{f}(x) = 0\text{ for all } x \in X \text{ and } f \in B. \] Therefore \( (Tf)^\ast = 0 \text{ for all } f \in B, \) and since \( B \) is semisimple, \( Tf = 0 \text{ for all } f \in B, \) as claimed.

We remark that if the maximal ideal space \( X \) of \( B \) has isolated points, then there exist nonzero compact multipliers of \( B. \) For, if \( x_0 \) is an isolated point of \( X, \) then by Silov’s Idempotent Theorem, there is an idempotent \( E \) in \( B \) satisfying \( E(x) = 1 \) if and only if \( x = x_0. \) Then the operator \( T \) defined by \( Tf = Ef = \hat{f}(x_0)E \) is clearly a nonzero multiplier which is compact since its range is one-dimensional.

Finally we remark that if \( H \) denotes the Hilbert space of square summable sequences with component-wise multiplication, then \( H \) is a commutative semisimple Banach algebra with discrete maximal ideal space. If we let \( (a_n) \) be a sequence of complex numbers converging to 0, then the operator \( T: \{x_n\} \to \{a_nx_n\} \) is a nonzero compact multiplier of \( H \) and \( \sigma(T) = \{a_n|n \text{ is a positive integer}\} \cup \{0\}. \)

**Bibliography**


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