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ON COMPACT MULTIPLIERS OF BANACH ALGEBRAS

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ABSTRACT. We show that if the maximal ideal space of a commutative semisimple Banach algebra B contains no isolated points, then every compact multiplier is trivial.

In [1] and [2] it was shown that if a commutative semisimple Banach algebra B satisfies certain regularity conditions and if the maximal ideal space of B contains no isolated points, then every compact multiplier of B is trivial. (T is a *multiplier* of B if T is a linear operator satisfying $T(fg) = f \cdot Tg$ for $f, g \in B$.) In this note we show that the regularity conditions used in [1] and [2] are unnecessary. Specifically we prove the following.

THEOREM. *Let B be a commutative semisimple Banach algebra and T a compact multiplier of B. If the maximal ideal space of B contains no isolated points, then* $T = 0$.

PROOF. Let X denote the maximal ideal space of B and assume that X contains no isolated points. Since T is a multiplier of B, there exists a complex-valued continuous function u on X with $(Tf)\hat{}(x) = u(x)\hat{f}(x)$ for all $f \in B$ and $x \in X$. We will show first that for each $x \in X$, $u(x)$ is an eigenvalue of the adjoint T^* of T. Indeed, for $x \in X$, let e_x denote the linear functional in B^* which is evaluation at x, i.e. $e_x(f) = \hat{f}(x)$. Then for each $f \in B$, we have $(T^*e_x)(f) = e_x(Tf) = (Tf)\hat{}(x) = u(x)\hat{f}(x) = u(x)e_x(f)$. Thus $T^*e_x = u(x)e_x$ which proves that $u(x)$ is an eigenvalue of T^*.

Now T, and hence T^*, is compact, so that the spectrum of T^*, $\sigma(T^*)$, is a denumerable set with 0 as its only possible limit point. $\sigma(T^*)$ also has the property that every nonzero element in $\sigma(T^*)$ is an eigenvalue of finite multiplicity. Suppose x_0 is a point in X which is not an isolated point. We claim that $u(x_0) = 0$. Indeed, suppose $u(x_0) \neq 0$. Since u is a continuous function on X and x_0 is a limit point of X, for each positive integer n, there exists an element x_n, $x_0 \neq x_n \in X$, with $|u(x_n) - (x_0)| < 1/n$. However, each nonzero eigenvalue of T^* has finite multiplicity and so $u(x_n) = u(x_0)$ for only finitely many n. Therefore $u(x_0)$ is a limit point of $\{u(x_n)\} \subset \sigma(T^*)$. However 0 is the only possible limit point of $\sigma(T^*)$ since T^* is compact. This contradiction shows that $u(x_0) = 0$. Since no element in X is an isolated point, by hypothesis, we conclude that $u(x) = 0$ for all $x \in X$. Hence

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$(Tf)\hat{}(x) = u(x)\hat{f}(x) = 0$ for all $x \in X$ and $f \in B$. Therefore $(Tf)\hat{} = 0$ for all $f \in B$, and since B is semisimple, $Tf = 0$ for all $f \in B$, as claimed.

We remark that if the maximal ideal space X of B has isolated points, then there exist nonzero compact multipliers of B. For, if x_0 is an isolated point of X, then by Šilov's Idempotent Theorem, there is an idempotent E in B satisfying $\hat{E}(x) = 1$ if and only if $x = x_0$. Then the operator T defined by $Tf = Ef = \hat{f}(x_0)E$ is clearly a nonzero multiplier which is compact since its range is one-dimensional.

Finally we remark that if H denotes the Hilbert space of square summable sequences with component-wise multiplication, then H is a commutative semisimple Banach algebra with discrete maximal ideal space. If we let $\{a_n\}$ be a sequence of complex numbers converging to 0, then the operator T: $\{x_n\} \to \{a_n x_n\}$ is a nonzero compact multiplier of H and $\sigma(T) = \{a_n | n \text{ is a positive integer}\} \cup \{0\}$.

Bibliography

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