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ON THE WEAK RADON-NIKODYM PROPERTY

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ABSTRACT. It is shown that in a Banach lattice the notions of the Radon-Nikodym property and the weak Randon-Nikodym property coincide. A problem of H. P. Lotz is solved affirmatively, namely, if the dual Y^* of a Banach space Y is complemented in a Banach lattice E and if l^1 does not embed into Y then Y^* has the Radon-Nikodym property.

In **[10]** Musial introduced the notion of the weak Radon-Nikodym property. In this paper we are going to show that in a Banach lattice the notions of the weak Radon-Nikodym property and the Radon-Nikodym property coincide and that if Y^* is the dual of a Banach space Y not containing l^1 and Y^* is complemented in a Banach lattice E then Y^* has the Radon-Nikodym property. This solves affirmatively a conjecture of H. P. Lotz **[9]**.

DEFINITION 1. Let E be a Banach space and let (T, Σ, λ) be a probability space. A function $f: T \to E$ is Pettis-integrable if

(i) for every $x^* \in E^*$ the map $t \to x^*f(t)$ is λ-measurable and λ-integrable and

(ii) for every $A \in \Sigma$ there is x_A in E such that $x^*(x_A) = \int_A x^*f(t)\, d\lambda$ for every x^* in E^*. In this case we write $x_A = \text{Pettis-}\int_A f\, d\lambda$.

DEFINITION 2. Let E be a Banach space and let (T, Σ, λ) be a probability space. A function $f: T \to E$ is Bochner-integrable if there exists a sequence (f_n) of simple functions such that

(i) $\lim_n \|f(t) - f_n(t)\| = 0$ for λ-almost all $t \in T$ and

(ii) $\lim_n \int_T \|f - f_n\|\, d\lambda = 0$.

It is easy to see that one can define

$$\text{Bochner-}\int_A f\, dP = \lim_n \int_A f_n\, d\lambda$$

for each $A \in \Sigma$. This definition is independent of the choice of the sequence (f_n). For more details see **[1**, Chapter II**]**.

DEFINITION 3. A Banach space has the Radon-Nikodym property (RNP) (resp. the weak Radon-Nikodym property (WRNP)) if for every complete probability space (T, Σ, λ) and every vector measure $m: \Sigma \to E$ such that $\|m(A)\| \leq \lambda(A)$ for every $A \in \Sigma$ there exists a Bochner-integrable (resp. Pettis-integrable) function f: $T \to E$ such that $m(A) = \text{Bochner-}\int_A f\, d\lambda$ (resp. $m(A) = \text{Pettis-}\int_A f\, d\lambda$) for every $A \in \Sigma$.

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There are Banach spaces with the WRNP and without the RNP and contrary to the RNP the WRNP is not hereditary with respect to subspaces **[10]**. However we have the following.

PROPOSITION 4. *Let X be a Banach space having the WRNP. Then neither c_0 nor L^1 can be embedded in X.*

PROOF. Suppose that c_0 embeds in X and let $T: c_0 \to X$ be this embedding. Let Σ be the σ-algebra of all Lebesgue measurable subsets of [0, 1] and λ be the Lebesgue measure.

Define a vector measure $G: \Sigma \to c_0$ by

$$G(A) = \left(\int_A \sin(2^n \pi t)\, d\lambda(t)\right).$$

According to the Riemann-Lebesgue lemma, the measure G has its values in c_0 and one has that $\|G(A)\| \leqslant \lambda(A)$ for every A in Σ. Define now a vector measure F: $\Sigma \to X$ by $F(A) = T(G(A))$ for every A in Σ. Hence $\|F(A)\| \leqslant \|T\|\lambda(A)$ for every A in Σ. Since X has the WRNP there exists f: $[0, 1] \to X$ Pettis-integrable such that

$$F(A) = \text{Pettis-}\int_A f\, d\lambda.$$

An appeal to **[3**, 3J] shows that the set $K = \{F(A) = T(G(A));\ A \in \Sigma\}$ is relatively compact in X, therefore the set $H = \{G(A);\ A \in \Sigma\}$ is relatively compact in c_0 because T is an isomorphism, so we will have a contradiction if we can prove that H is not relatively compact in c_0.

Suppose that H is relatively compact in c_0. Using a well-known characterization of compact subsets of c_0 we see that

$$\lim_n \sup_{A \in \Sigma} \left|\int_A \sin(2^n \pi t)\, d\lambda(t)\right| = 0, \tag{$*$}$$

but

$$a_n = \sup_{A \in \Sigma} \left|\int_A \sin(2^n \pi t)\, d(t)\right| \geqslant \frac{1}{2}\int_0^1 |\sin(2^n \pi t)|\, d\lambda(t).$$

Let $A_n = \{t \in [0, 1];\ |\sin(2^n \pi t)| \geqslant 1/\sqrt{2}\}$; then $\lambda(A_n) = \frac{1}{4}$, hence $a_n \geqslant (1/2)(1/\sqrt{2})(1/4) = 1/8\sqrt{2}$ and this contradicts ($*$).

For the L^1 part, use the same idea by considering $G(A) = 1_A$ and the well-known fact that $\{G(A);\ A \in \Sigma\}$ is not relatively compact in L^1 **[2**, p. 261].

It was pointed out to us by the referee that the fact that c_0 cannot be embedded in a Banach space with the WRNP was also obtained by Janicka **[7]** then by Musial **[11]** using martingales.

We shall say that a Banach space E has the separable complementation property if for every separable subspace Z in E there exists a separable subspace Y containing Z and complemented in E.

The fact that c_0 cannot be embedded in a space having the WRNP is essential in the following theorem.

THEOREM 5. *Let E be a Banach lattice. If E has the WRNP then E has the RNP.*

PROOF. By Proposition 2, the space c_0 cannot be embedded in E and therefore E is weakly sequentially complete [8, p. 37], and hence E is an order continuous Banach lattice. By Kakutani's theorem [8, p. 9] there is a family (possibly uncountable) of complemented subspaces X_α of E, each of them having a weak order unit such that $E = \Sigma_\alpha X_\alpha$. In addition if Z is a separable subspace of X, then an α_0 can be chosen so that X_{α_0} contains Z. Let Y be a separable subspace of X_{α_0} containing Z and complemented in X_{α_0} (this can be done because X_{α_0} is WCG); this means that Y is complemented in E, hence E has the separable complementation property. Apply [10, Theorem 1] to finish the proof.

In the proof of the above theorem we saw that an order continuous Banach lattice has the separable complementation property. This fact will be used later.

In [12] Musial and Ryll-Nardzewski proved the following theorem by a different method.

THEOREM 6 (MUSIAL AND RYLL-NARDZEWSKI). *If the dual X^* of a Banach space X has the WRNP, then l^1 does not embed into X.*

PROOF. If l^1 embeds into X, then by [4, Proposition 1], L^1 embeds into X^*, but this is impossible by Proposition 4.

The converse of Theorem 6 is true and is due to Janicka [7] and Bourgain (unpublished). It is a consequence of [6] and [14]. Combining this with Theorem 5 yields the following result of H. P. Lotz [9], [1].

THEOREM 7 (LOTZ). *Let E be a Banach lattice; then E^* has the RNP if and only if l^1 does not embed into E.*

PROPOSITION 8. *Let X be a Banach space complemented in a Banach lattice E. If X has the WRNP, then X has the RNP.*

PROOF. If X does have the WRNP, then c_0 does not embed in X; by [8, p. 36] X is isomorphic to a subspace of an order continuous Banach lattice. Therefore X has the RNP.

The following proposition answers positively a problem of H. P. Lotz [9, Problem 4].

PROPOSITION 9. *Let X^* be the dual of a Banach space X not containing a subspace isomorphic to l^1 and suppose that X^* is complemented in a Banach lattice E. Then X^* has the RNP.*

PROOF. By [7], the space X^* has the WRNP. Apply Proposition 8 to finish the proof.

In [13] Pełczynski proved under a special assumption which was later removed by Hagler [5] that if L^1 does embed in E^* then l^1 embeds into E.

The following theorem gives a proof of the above result via the WRNP.

THEOREM 10. *If L^1 does embed in the dual E^* of a Banach space E, then l^1 does embed in E.*

PROOF. If L^1 does embed in E^* then E^* will not have the WRNP (Proposition 4) and therefore l^1 embeds into E.

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