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AN ELEMENTARY PROOF OF THE STONE-WEIERSTRASS THEOREM

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ABSTRACT. An elementary proof of the Stone-Weierstrass theorem is given.

In this note we give an elementary proof of the Stone-Weierstrass theorem. The proof depends only on the *definitions* of compactness ("each open cover has a finite subcover") and continuity ("the inverse images of open sets are open"), two simple consequences of these definitions (i.e. "a closed subset of a compact space is compact," and "a positive continuous function on a compact set has a positive infimum"), and the elementary Bernoulli inequality:

$$(1 + h)^n \geqslant 1 + nh \qquad (n = 1, 2, \dots)$$

if $h \geqslant -1$.

In the beautiful and elementary proof of the classical Weierstrass theorem given by Kuhn [1], it is observed that it suffices to be able to approximate, by polynomials, the step function which is 1 on the interval $[0, \frac{1}{2})$ and 0 on the interval $[\frac{1}{2}, 1]$ uniformly *outside* of each neighborhood of $\frac{1}{2}$. The main step in our proof (Lemma 2) is the general analogue of this. It shows that it suffices to be able to approximate, by elements of the subalgebra, a given "generalized step function" (i.e. a function which is 0 on a closed set and 1 off the set) uniformly on the closed set and off a neighborhood of this set.

It should be remarked that when our proof is specialized to the classical case of polynomials in $C[a, b]$, it is even "simpler" than Kuhn's proof in the sense that no "change of variables" argument is necessary, nor is it necessary to appeal to the fact that continuous functions on $[a, b]$ are uniformly continuous. (Kuhn's proof also used the Bernoulli inequality.)

In particular, it is perhaps worth emphasizing that, in contrast to many proofs of the Stone-Weierstrass theorem, we do *not* appeal to any of the following facts:

(a) the classical Weierstrass theorem (nor even the special case of uniformly approximating $f(t) = |t|$ on $[-1, 1]$ by polynomials);

(b) that the closure of a subalgebra is a subalgebra;

(c) that the closure of a subalgebra is a sublattice.

Let T be a compact topological space and $C(T)$ the set of all real-valued continuous functions on T. A neighborhood of a point in T is an open set which

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contains the point. Let $\mathfrak{A}$ be a subset of $C(T)$ with the properties:

(i) x, y in $\mathfrak{A}$, α, β in $\mathbf{R}$ implies $\alpha x + \beta y \in \mathfrak{A}$;

(ii) x, y in $\mathfrak{A}$ implies $x \cdot y \in \mathfrak{A}$;

(iii) $1 \in \mathfrak{A}$;

(iv) if t_1 and t_2 are distinct points in T, then there exists $x \in \mathfrak{A}$ such that $x(t_1) \neq x(t_2)$.

In other words, $\mathfrak{A}$ is a "subalgebra of $C(T)$ which contains constants and separates points."

The Stone-Weierstrass theorem may be stated as follows. *If $\mathfrak{A}$ is a subalgebra of $C(T)$ which contains constants and separates points, then the elements of $C(T)$ can be uniformly approximated by the elements of $\mathfrak{A}$.* That is, given $f \in C(T)$ and $\varepsilon > 0$ there exists $g \in \mathfrak{A}$ such that $\sup_{t \in T} |f(t) - g(t)| < \varepsilon$.

It is convenient to divide the proof into three steps. The essential step is Lemma 1. For brevity, the norm notation $\|x\| = \sup\{|x(t)| \mid t \in T\}$ will sometimes be used.

LEMMA 1. *Let $t_0 \in T$ and let U be a neighborhood of t_0. Then there is a neighborhood $V = V(t_0)$ of t_0, $V \subset U$, with the following property. For each $\varepsilon > 0$, there exists $x \in \mathfrak{A}$ such that*

(1) $0 \leqslant x(t) \leqslant 1$, $t \in T$;

(2) $x(t) < \varepsilon$, $t \in V$;

(3) $x(t) > 1 - \varepsilon$, $t \in T \setminus U$.

PROOF. For each $t \in T \setminus U$, the point separating property (iv) implies that there is a function $g_t \in \mathfrak{A}$ with $g_t(t) \neq g_t(t_0)$. Then the function $h_t = g_t - g_t(t_0) \cdot 1$ is in $\mathfrak{A}$ and $h_t(t) \neq h_t(t_0) = 0$. The function $p_t = (1/\|h_t^2\|)h_t^2$ is in $\mathfrak{A}$, $p_t(t_0) = 0$, $p_t(t) > 0$, and $0 \leqslant p_t \leqslant 1$.

Let $U(t) = \{s \in T \mid p_t(s) > 0\}$. Then $U(t)$ is a neighborhood of t. By compactness of $T \setminus U$, there exist a finite number of points $\{t_1, t_2, \ldots, t_m\}$ in $T \setminus U$ such that $T \setminus U \subset \bigcup_1^m U(t_i)$. Let $p = (1/m)\sum_1^m p_{t_i}$. Then $p \in \mathfrak{A}$, $0 \leqslant p \leqslant 1$, $p(t_0) = 0$, and $p > 0$ on $T \setminus U$.

Again using the compactness of $T \setminus U$, there exists $0 < \delta < 1$ such that $p \geqslant \delta$ on $T \setminus U$. Let $V = \{t \in T \mid p(t) < \delta/2\}$. Then V is a neighborhood of t_0 and $V \subset U$.

Let k be the smallest integer which is greater than $1/\delta$. Then $k - 1 \leqslant 1/\delta$ which implies that $k \leqslant (1 + \delta)/\delta < 2/\delta$. Thus $1 < k\delta < 2$. Consider the functions q_n defined by

$$q_n(t) = \left[1 - p^n(t)\right]^{k^n} \qquad (n = 1, 2, \ldots).$$

Clearly, $q_n \in \mathfrak{A}$, $0 \leqslant q_n \leqslant 1$, and $q_n(t_0) = 1$. For each $t \in V$, $kp(t) \leqslant k\delta/2 < 1$ so that, by Bernoulli's inequality,

$$q_n(t) \geqslant 1 - \left[kp(t)\right]^n \geqslant 1 - (k\delta/2)^n \to 1$$

uniformly on V. For $t \in T \setminus U$, $kp(t) \geqslant k\delta > 1$ and, using Bernoulli's inequality,

$$q_n(t) = \frac{1}{k^n p^n(t)}\left[1 - p^n(t)\right]^{k^n} k^n p^n(t) \leqslant \frac{1}{\left[kp(t)\right]^n}\left[1 - p^n(t)\right]^{k^n}\left[1 + k^n p^n(t)\right]$$

$$\leqslant \frac{1}{\left[kp(t)\right]^n}\left[1 - p^n(t)\right]^{k^n}\left[1 + p^n(t)\right]^{k^n} = \frac{1}{\left[kp(t)\right]^n}\left[1 - p^{2n}(t)\right]^{k^n}$$

$$\leqslant \frac{1}{(k\delta)^n} \to 0$$

uniformly on $T \setminus U$.

Thus for n sufficiently large, the function q_n has the property that $0 \leqslant q_n \leqslant 1$, $q_n < \varepsilon$ on $T \setminus U$, and $q_n > 1 - \varepsilon$ on V. The result follows by taking $x = 1 - q_n$. □

Lemma 2. *Let A and B be disjoint closed subsets of T. Then for each $0 < \varepsilon < 1$, there exists $x \in \mathfrak{A}$ such that*

(1) $0 \leqslant x(t) \leqslant 1, t \in T$;

(2) $x(t) < \varepsilon, t \in A$;

(3) $x(t) > 1 - \varepsilon, t \in B$.

Proof. Let $U = T \setminus B$. For each $t \in A$, choose the neighborhood $V(t)$ of t as in Lemma 1. Then there exists a finite set of points $\{t_1, t_2, \ldots, t_m\}$ in A such that $A \subset \bigcup_1^m V(t_i)$. By the choice of $V(t_i)$, there exist $x_i \in \mathfrak{A}$ $(i = 1, 2, \ldots, m)$ with $0 \leqslant x_i \leqslant 1$, $x_i < \varepsilon/m$ on $V(t_i)$, and $x_i > 1 - \varepsilon/m$ on $T \setminus U = B$. Then the function $x = x_1 \cdot x_2 \cdots x_m$ is in $\mathfrak{A}$, $0 \leqslant x \leqslant 1$, $x < \varepsilon/m \leqslant \varepsilon$ on $\bigcup_1^m V(t_i) \supset A$, and (using Bernoulli's inequality) $x > (1 - \varepsilon/m)^m \geqslant 1 - \varepsilon$ on B. □

Finally, we turn to the proof of the Stone-Weierstrass theorem. Let $f \in C(T)$ and $\varepsilon > 0$. To complete the proof, it suffices to show the existence of a $g \in \mathfrak{A}$ such that

$$|f(t) - g(t)| < 2\varepsilon, \qquad t \in T. \tag{$*$}$$

By replacing f by $f + \|f\|$, we can assume that $f \geqslant 0$. We may also assume that $\varepsilon < \frac{1}{3}$. Choose an integer n so that $(n-1)\varepsilon \geqslant \|f\|$. Define the sets A_j, B_j $(j = 0, 1, \ldots, n)$ by

$$A_j = \left\{t \in T \,|\, f(t) \leqslant \left(j - \tfrac{1}{3}\right)\varepsilon\right\}, \qquad B_j = \left\{t \in T \,|\, f(t) \geqslant \left(j + \tfrac{1}{3}\right)\varepsilon\right\}.$$

Note that A_j and B_j are disjoint closed sets in T, $\varnothing = A_0 \subset A_1 \subset \cdots \subset A_n = T$, and $B_0 \supset B_1 \supset \cdots \supset B_n = \varnothing$. For each $j = 0, 1, \ldots, n$, Lemma 2 implies that there is $x_j \in \mathfrak{A}$, with $0 \leqslant x_j \leqslant 1$, $x_j < \varepsilon/n$ on A_j, and $x_j > 1 - \varepsilon/n$ on B_j.

Then the function $g = \varepsilon \sum_0^n x_i$ is in $\mathfrak{A}$. For any $t \in T$, we have $t \in A_j \setminus A_{j-1}$ for some $j \geqslant 1$ which implies that

$$\left(j - \tfrac{4}{3}\right)\varepsilon < f(t) \leqslant \left(j - \tfrac{1}{3}\right)\varepsilon \tag{$**$}$$

and

$$x_i(t) < \varepsilon/n \quad \text{for every } i \geqslant j. \tag{$***$}$$

Also, $t \in B_i$ for every $i \leqslant j - 2$ which implies

$$x_i(t) > 1 - \varepsilon/n \quad \text{for every } i \leqslant j - 2. \tag{$****$}$$

Using (***), we obtain

$$\begin{aligned} g(t) &= \varepsilon \sum_{0}^{j-1} x_i(t) + \varepsilon \sum_{j}^{n} x_i(t) \\ &\leqslant j\varepsilon + \varepsilon(n - j + 1)\varepsilon/n \leqslant j\varepsilon + \varepsilon^2 < \left(j + \tfrac{1}{3}\right)\varepsilon. \end{aligned}$$

Using (****), we obtain for $j \geqslant 2$

$$\begin{aligned} g(t) &\geqslant \varepsilon \sum_{0}^{j-2} x_i(t) \geqslant (j - 1)\varepsilon(1 - \varepsilon/n) \\ &= (j - 1)\varepsilon - ((j - 1)/n)\varepsilon^2 > (j - 1)\varepsilon - \varepsilon^2 > \left(j - \tfrac{4}{3}\right)\varepsilon. \end{aligned}$$

The inequality $g(t) > (j - \frac{4}{3})\varepsilon$ is trivially true for $j = 1$. Thus

$$|f(t) - g(t)| \leqslant \left(j + \tfrac{1}{3}\right)\varepsilon - \left(j - \tfrac{4}{3}\right)\varepsilon < 2\varepsilon. \quad \square$$

References

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