PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 1, January 1981

ANOTHER q-EXTENSION OF THE BETA FUNCTION

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ABSTRACT. Another q-extension of the beta function is given. This one has a special case that is a symmetric extension of the symmetric beta distribution.

1. Introduction. The beta function

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\, dt \tag{1.1}$$

was reduced to a function of one variable by Euler when he proved that

$$B(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta). \tag{1.2}$$

Changes of variable can be made in (1), so other integrals can be evaluated. One trivial change is to

$$\int_{-c}^{d}\left(1-\frac{x}{d}\right)^{\beta-1}\left(1+\frac{x}{c}\right)^{\alpha-1} dx = \frac{(c+d)^{\alpha+\beta-1}}{c^{\alpha-1}d^{\beta-1}}\,\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{1.3}$$

The special case $\alpha = \beta$, $c = d$ is important, for the resulting distribution function is the weight function for the ultraspherical polynomials, which contain the zonal spherical harmonics (or zonal functions) on the k-dimensional sphere.

Recently there has been a renewal of interest in basic hypergeometric, or q-series, extensions of classical results. If $0 < q < 1$, define

$$(a; q)_\infty = \prod_{n=0}^{\infty}(1 - aq^n), \tag{1.4}$$

$$(a; q)_n = (a; q)_\infty / (aq^n; q)_\infty \tag{1.5}$$

and

$$\Gamma_q(x) = ((q; q)_\infty / (q^x; q)_\infty)(1-q)^{1-x}. \tag{1.6}$$

The q-binomial theorem [**1**, Theorem 2.1]

$$\frac{(ax; q)_\infty}{(x; q)_\infty} = \sum_{n=0}^{\infty}\frac{(a; q)_n}{(q; q)_n}x^n \tag{1.7}$$

can be rewritten in a form which extends (1.1). Following F. H. Jackson define

$$\int_0^d f(x)\, d_qx = d(1-q)\sum_{n=0}^{\infty} f(dq^n)q^n. \tag{1.8}$$

Received by the editors February 21, 1980.
1980 *Mathematics Subject Classification*. Primary 33A15, 33A35.
Key words and phrases. Beta function, q-binomial theorem, basic hypergeometric functions.
[1]Supported in part by NSF grant MCS-75-19162.
[2]Supported in part by NSF grant MCS-78-07244.

0002-9939/81/0000-0021/$02.00

Then (1.7) is essentially equivalent to

$$\int_0^1 t^{\alpha-1}\frac{(tq;q)_\infty}{(tq^\beta;q)_\infty}\,d_qt=\frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}. \tag{1.9}$$

One drawback with the q-integral is that changes of variable are usually not possible. However there are often extensions of the integrals that arise after a change of variable. Two extensions of the beta function put on $(0, \infty)$ as

$$\int_0^\infty \frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}}\,dt=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{1.10}$$

were found by Ramanujan. This was pointed out in [3] and simple proofs were given of the following.

$$\int_0^\infty t^{\alpha-1}\frac{(-tq^{\alpha+\beta};q)_\infty}{(-t;q)_\infty}\,dt=\frac{\Gamma(\alpha)\Gamma(1-\alpha)\Gamma_q(\beta)}{\Gamma_q(1-\alpha)\Gamma_q(\alpha+\beta)} \tag{1.11}$$

and

$$\int_0^\infty t^{\alpha-1}\frac{(-ctq^{\alpha+\beta};q)_\infty\,d_qt}{(-ct;q)_\infty}=\frac{\Gamma_q(\alpha)\Gamma_q(\beta)(-cq^\alpha;q)_\infty(-q^{1-\alpha}/c;q)_\infty}{\Gamma_q(\alpha+\beta)(-c;q)_\infty(-q/c;q)_\infty}. \tag{1.12}$$

However none of these integrals contains a special case that extends the symmetric beta function in a way so that the symmetry is obvious. We were led to such an extension while studying some orthogonal polynomials [2]. This extension is of independent interest, and so will be given here.

2. An extension of the beta function.

THEOREM 1. *If* $|q| < 1$ *and there are no zero factors in the denominator of the integrals, then*

$$\int_{-c}^{d}\frac{(-qx/c;q)_\infty(qx/d;q)_\infty}{(-ax/c;q)_\infty(bx/d;q)_\infty}\,d_qx$$
$$=\frac{(1-q)(q;q)_\infty(ab;q)_\infty cd(-c/d;q)_\infty(-d/c;q)_\infty}{(a;q)_\infty(b;q)_\infty(c+d)(-bc/d;q)_\infty(-ad/c;q)_\infty}, \tag{2.1}$$

or, when $0 < q < 1$,

$$\int_{-c}^{d}\frac{(-qx/c;q)_\infty(qx/d;q)_\infty}{(-xq^\alpha/c;q)_\infty(xq^\beta/d;q)_\infty}\,d_qx$$
$$=\frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}\,\frac{cd}{c+d}\,\frac{(-c/d;q)_\infty(-d/c;q)_\infty}{(-q^\beta c/d;q)_\infty(-q^\alpha d/c;q)_\infty}. \tag{2.2}$$

PROOF. As with the usual integral, $\int_{-c}^{d} = \int_0^d - \int_0^{-c}$. Thus if I denotes the integral in (2.1)

$$\begin{aligned} I &= d(1-q)\sum_{n=0}^{\infty}\frac{(-dq^{n+1}/c;q)_\infty(q^{n+1};q)_\infty q^n}{(-adq^n/c;q)_\infty(bq^n;q)_\infty} \\ &\quad + c(1-q)\sum_{n=0}^{\infty}\frac{(q^{n+1};q)_\infty(-cq^{n+1}/d;q)_\infty q^n}{(aq^n;q)_\infty(-bcq^n/d;q)_\infty} \\ &= d(1-q)\frac{(-dq/c;q)_\infty(q;q)_\infty}{(-ad/c;q)_\infty(b;q)_\infty}\,{}_2\varphi_1\left(\begin{matrix}-ad/c,\ bd\\ -dq/c\end{matrix};q,q\right) \\ &\quad + c(1-q)\frac{(-cq/d;q)_\infty(q;q)_\infty}{(a;q)_\infty(-bc/d;q)_\infty}\,{}_2\varphi_1\left(\begin{matrix}-bc/d,\ a\\ -cq/d\end{matrix};q,q\right) \end{aligned}$$

where

$${}_2\varphi_1\left(\begin{matrix}a,\ b\\ c\end{matrix};q,x\right) = \sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n}x^n. \tag{2.3}$$

It is not possible to sum these series separately (in the general case), for if it could be done we could evaluate $\int_0^1(1-t)^{\alpha-1}(1+t)^{\beta-1}\,dt$. So if there is to be a hope of evaluating (2.1) the two series must be put together into a single sum. To do this recall a transformation of Heine [1, Corollary 2.3].

$${}_2\varphi_1\left(\begin{matrix}a,\ b\\ c\end{matrix};q,x\right) = \frac{(ax;q)_\infty(b;q)_\infty}{(x;q)_\infty(c;q)_\infty}\,{}_2\varphi_1\left(\begin{matrix}c/b,\ x\\ ax\end{matrix};q,b\right). \tag{2.4}$$

Use of (2.4) gives

$$\begin{aligned} I &= \frac{d(1-q)}{(1-b)}\sum_{n=0}^{\infty}\frac{(q/a;q)_n}{(bq;q)_n}\left(\frac{-ad}{c}\right)^n \\ &\quad + \frac{c(1-q)}{(1-a)}\sum_{n=0}^{\infty}\frac{(q/b;q)_n}{(aq;q)_n}\left(\frac{-bc}{d}\right)^n. \end{aligned}$$

Replace n by $-n-1$ in the second sum and use

$$\frac{(a;q)_{-n}}{(b;q)_{-n}} = \frac{(bq^{-n};q)_n}{(aq^{-n};q)_n} = \left(\frac{b}{a}\right)^n\frac{(q/b;q)_n}{(q/a;q)_n}$$

to obtain

$$I = \frac{d(1-q)}{(1-b)}\sum_{-\infty}^{\infty}\frac{(qa^{-1};q)_n}{(bq;q)_n}\left(\frac{-ad}{c}\right)^n.$$

Ramanujan summed this series as

$$\sum_{-\infty}^{\infty}\frac{(a;q)_n}{(b;q)_n}x^n = \frac{(ax;q)_\infty(q/ax;q)_\infty(q;q)_\infty(b/a;q)_\infty}{(x;q)_\infty(b/ax;q)_\infty(b;q)_\infty(q/a;q)_\infty}. \tag{2.5}$$

This is another way of writing (1.12). Using this gives

$$I = \frac{d(1-q)(-qd/c;q)_\infty(-c/d;q)_\infty(q;q)_\infty(ab;q)_\infty}{(-ad/c;q)_\infty(-bc/d;q)_\infty(a;q)_\infty(b;q)_\infty}.$$

This derivation used the conditions $|b| < |d/c| < 1/|a|$ to be able to sum (2.5). However both sides of (2.1) are meromorphic functions with at most poles of finite order at any finite point as functions of either a or b, and as functions of c and d when c and d are in any compact subset of the plane with the origin removed. Use of analytic continuation completes the proof of Theorem 1.

The referee asked if a q-extension of (1.3) exists as a Riemann integral. It does, and the most general one with a real absolutely continuous integrand known at present can be written as

$$\frac{1}{\pi}\int_{-1}^{1}\frac{h_q(1;x)h_q(q^{1/2};x)h_q(-1;x)h_q(-q^{1/2};x)}{h_q(q^\alpha;x)h_q(q^\beta;x)h_q(-q^\gamma;x)h_q(-q^\delta;x)}\frac{dx}{(1-x^2)^{1/2}}$$
$$=\frac{\Gamma_q(\alpha+\beta)\Gamma_q(\gamma+\delta)(-1;q)_\infty(-q^{1/2};q)_\infty^2(-q;q)_\infty}{\Gamma_q(\alpha+\beta+\gamma+\delta)\left[\Gamma_q\left(\frac{1}{2}\right)\right]^2(-q^{\alpha+\gamma};q)_\infty(-q^{\alpha+\delta};q)_\infty(-q^{\beta+\gamma};q)_\infty(-q^{\beta+\delta};q)_\infty}$$

when $0<q<1$, $\alpha,\beta,\gamma,\delta>0$ and $h_q(a;x)=\prod_{n=0}^{\infty}(1-2axq^n+a^2q^{2n})$. This will appear in [**4**].

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