PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 1, January 1981

SIMPLE HOMOTOPY TYPES FOR (G, m)-COMPLEXES

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ABSTRACT. Let G be a finite group. We use the fact that each element of the Whitehead group $\mathrm{Wh}(G)$ may be represented by at most a 2×2 (nonsingular) matrix to deduce results about when simple homotopy type and homotopy type agree. As examples, we give complete descriptions of the simple homotopy types for $(Z_m \times Z_n, 2)$-complexes, provided $SK_1(Z(Z_m \times Z_n)) = 0$.

1. Representing elements of $\mathrm{Wh}(G)$ **by matrices of small dimension.** Let G be a group and ZG be the integral group ring of G. The group $K_1(ZG)$ is defined to be the abelianization of the infinite general linear group $\mathrm{GL}(ZG) = \lim \mathrm{GL}(ZG, n)$. The Whitehead group $\mathrm{Wh}(G)$ of G is $K_1(ZG)/\{\pm G\}$.

DEFINITION. We say that the representing dimension of G is $\leqslant i$ ($r - \dim G \leqslant i$) if and only if each element of $\mathrm{Wh}(G)$ can be represented by a matrix in $\mathrm{GL}(ZG, i)$.

Let $U(R)$ denote the group of units in the ring R. Recall that $SK_1(ZG)$ is defined in general to be $\ker[K_1(ZG) \to K_1(QG)]$ [S, p. 208] and that, when G is abelian, $\mathrm{Wh}(G) \cong U(ZG)/\pm G \oplus SK_1(ZG)$ [C, p. 41]. Thus, when G is abelian, $r - \dim G = 1$ if and only if $SK_1(ZG) = 0$. Known results on $r - \dim G$ can now be summarized as follows.

THEOREM 1. (a) *If G is finite then* $r - \dim G \leqslant 2$ [**L**, *p.* 141], [**B**, *p.* 183].

(b) *If $G = Z_n$ (the cyclic group of order n) or if $G = Z_p \oplus Z_{p^i}$ for some prime p then* $r - \dim G = 1$ [**L**, *Theorem* 1.1, *p.* 130].

(c) *If G is a finite abelian group with a subgroup isomorphic to $Z_p \oplus Z_p \oplus Z_p$ (p an odd prime), or isomorphic to $Z_2 \oplus Z_2 \oplus Z_2$ or to $Z_4 \oplus Z_4$, then* $r - \dim G = 2$ [**ADS**, *p.* 5].

At the present time it appears that no examples of finitely presentable groups G are known for which $r - \dim G > 2$. On the other hand, it is also unknown whether, given a finitely presentable group G, there necessarily exists an integer $n = n(G)$ such that $r - \dim G \leqslant n$.

2. Surjectivity of $\xi(X \vee iS^n) \to \mathrm{Wh}(\pi_1 X)$. Let X be a pointed topological space and denote by $\xi(X)$ the group of homotopy classes of pointed self-homotopy equivalences of $X \to X$. Let iS^n be the bouquet (one point union) of i-copies of the n-sphere S^n.

Received by the editors April 6, 1979 and, in revised form, September 29, 1979.
AMS (MOS) subject classifications (1970). Primary 55D15.

0002-9939/81/0000-0025/$02.25

THEOREM 2. *Let X be a pointed* CW-*complex and assume that* $r - \dim \pi_1 X \leq i$. *For any integer $n \geq 2$, the function $\xi(X \vee iS^n) \to \mathrm{Wh}(\pi_1 X)$ given by $[f] \to \tau(f)$ is surjective.*

PROOF. Let $C_*(\tilde{X})$ be the cellular chain complex of the universal cover $\tilde{X}$ of X and $((ZG)^i, n)$ the trivial chain complex having $(ZG)^i$ in dimension n and zero elsewhere. Then

$$C_*(X \vee iS^n) = C_*(\tilde{X}) \oplus (ZG^i, n) \qquad (G = \pi_1 X).$$

Represent $\tau \in \mathrm{Wh}(\pi_1 X)$ by an $(i \times i)$ invertible matrix M ($\tau = \{M\}$) and define

$$\tilde{M} = \mathrm{id} \oplus M^{(-1)^n}\colon C_*(\tilde{X}) \oplus (ZG^i, n) \to C_*(\tilde{X}) \oplus (ZG^i, n).$$

Then $\tilde{M}$ is realizable by a pointed self-homotopy equivalence $[m] \in \xi(X \vee iS^n)$ having torsion $\tau(m) = (-1)^{2n}\{M\} = \tau$. Indeed, m can be defined so that $m|X = 1$ and m maps each n-sphere of $X \vee iS^n$ so as to realize a row of M. □

COROLLARY 1. *Suppose $r - \dim G \leq i$ and let X and Y have a fundamental group isomorphic to G. Then $X \simeq Y$ implies that, for any $n \geq 2$, $X \vee iS^n \simeq_s Y \vee iS^n$.*

PROOF. This follows from [C, Theorem 24.4]. □

Question 1. Do there exist CW-complexes X, Y such that $X \vee 2S^n \simeq_s Y \vee 2S^n$ but $X \vee S^n \not\simeq_s Y \vee S^n$?

3. Simple homotopy types of (G, m)**-complexes.** A (G, m)-complex is a finite, connected m-dimensional CW-complex X with fundamental group isomorphic to G and trivial homotopy groups in dimensions between 1 and m. For example, any finite, connected 2-dimensional CW-complex is a $(\pi_1 X, 2)$-complex.

For a pair (G, m), let $\chi_{\min}(G, m) = \min\{(-1)^m \chi(X) | X \text{ is a } (G, m)\text{-complex}\}$. The *level* of a (G, m)-complex X is the number $l(X) = (-1)^m \chi(X) - \chi_{\min}(G, m)$.

THEOREM 3. *Let G be a finite group. Any two (G, m)-complexes at the same level $l \geq 2$ have the same simple homotopy type. If $r - \dim G = 1$ and $\chi_{\mathrm{mim}}(G, m) \geq 1$, then any two (G, m)-complexes at the same level $l \geq 1$ have the same simple homotopy type.*

PROOF. Let X and Y be (G, m)-complexes at level l and Z be a (G, m)-complex at level 0. By [$\mathbf{D}_2$, Theorem 1], if $l \geq 2$ then $X \simeq Y \simeq Z \vee lS^m$. By Theorem 2, any homotopy equivalence from X (or Y) $\to Z \vee lS^m$ can be converted to a simple homotopy by composing with an appropriate self-equivalence of $Z \vee lS^m$. To see this let $f\colon X \xrightarrow{\simeq} Z \vee lS^m$ and $[g] \in \xi(Z \vee lS^m)$ with $\tau(g) = \tau(f)$. Then a homotopy inverse $\bar{g}$ of g yields

$$\tau(\bar{g}f) = \tau(\bar{g}) + \bar{g}_*\tau(f) = -\bar{g}_*\tau(g) + \bar{g}_*\tau(f) = 0.$$

If $\chi_{\min}(G, m) \geq 1$, then $\pi_2 Z \oplus ZG$ has the Eichler condition [**DSII**, p. 124] and by a theorem of W. Browning [**Br**], $X \simeq Y \simeq Z \vee S^m$. Since $r - \dim G = 1$, another application of Theorem 2 finishes the proof. □

We observe that $\chi_{\min}(G, m) \geq 0$ for G finite and ≥ 1 for m even. In fact, a simple argument shows that $\chi_{\min}(G, m) = 0$ forces G to be a *periodic* group of *period* $m + 1$.

Question 2. For $(G, 2)$-complexes, is there a distinction between homotopy and simple homotopy type at levels 0 or 1?

The following special results are known.

EXAMPLE 1. For $G = Z_n$ and m even, homotopy and simple homotopy agree at all levels ([**D$_1$**, p. 279] and [**DSI**, p. 32]). [**DSI**] proves the case $m = 2$. The extension to $m > 2$ is similar.

EXAMPLE 2. For $G = Z_n$ ($n \neq 2, 3, 4, 6$) and $m = 2s - 1$ ($s > 2$), then at levels $l \geqslant 1$, the notions agree, while at level zero, there are $|Z_n^*/\pm(Z_n^*)^s|$ distinct homotopy types [**D$_1$**, p. 279] but the cardinality of the simple homotopy types at level zero is *countably infinite* [**W$_1$**, Theorem 14E3, p. 207].

EXAMPLE 3. If $G = D_n$, the dihedral group of order $2n$, then G is periodic if and only if n is odd. If n is odd, however, any two (G, m)-complexes at any level > 1 have the same homotopy type [**D$_2$**, Example 3], [**D$_1$**, 10.5]. Because the stabilization map $UZD_n \to K_1ZD_n$ is surjective [**JM**], we have $r - \dim D_n = 1$. Thus there is a single simple homotopy type at each level above zero. If n is even, then G is *not* periodic and $\chi_{\min}(G, m) \geqslant 1$. Again, things are trivial above level zero. *What happens at level zero*?

EXAMPLE 4. W. Metzler has recently shown [**M**] that the 2-complex K corresponding to the presentation $(a, b\colon b^m, aba^{-1}b^{-1})$ of $G = Z \oplus Z_m$ has the property that every unit of ZG can be realized as the torsion of a self-equivalence. Moreover, if m is prime no nonzero element of $SK_1(ZG)$ can be so realized, and it can occur that $SK_1(ZG) \neq 0$. (This gives the first known examples of 2-complexes K^2 for which $\tau\colon \xi(K) \to \mathrm{Wh}(\pi_1 K)$ is not onto.) This implies [**C**, 24.4] that there is a complex X with $X \simeq K^2$ and $X \not\simeq_s K^2$. But it is not known whether this X can be chosen to be 2-dimensional.

It is known that $SK_1(Z_m \times Z_n) = 0$ for certain m, n. This case is discussed in the next section.

4. $(Z_m \times Z_n, 2)$-complexes. Let $\pi = Z_m \times Z_n$ be the product of two finite cyclic groups with generators x, y of orders m and n, respectively, and let m divide n. Let X denote the $(\pi, 2)$-complex modeled on the presentation

$$\mathcal{P} = \{x, y\colon x^m, y^n, [x, y]\}$$

of π and let $\pi_2 = \pi_2 X$. By looking at the cellular chain complex $C_*(\tilde{X})$ of the universal cover $\tilde{X}$ of X, we may identify π_2 as the kernel of ∂_2 in the following sequence

$$C_*(\tilde{X}) : (Z\pi)^3 \xrightarrow[\partial_2]{\begin{bmatrix} N_x & 1-y & 0 \\ 0 & x-1 & N_y \end{bmatrix}} (Z\pi)^2 \xrightarrow[\partial_1]{(x-1, y-1)} Z\pi \overset{\varepsilon}{\twoheadrightarrow} Z \qquad (*)$$

where $N_x = 1 + x + \cdots + x^{m-1}$ and $N_y = 1 + y + \cdots + y^{n-1}$ and ε is the augmentation homomorphism.

In this section we show that for $SK_1(Z\pi) = 0$, the natural map τ: $\xi X \to \text{Wh}(\pi)$ ($[f] \mapsto \tau(f)$, the Whitehead torsion of f) from the group ξX of homotopy classes of (pointed) self-homotopy equivalences of the model X to the Whitehead group $\text{Wh}(\pi)$ is surjective.

THEOREM 4. *Let $\pi = Z_m \times Z_n$ with generators x and y (respectively) and X be as given above. Write $\text{Wh}(\pi) \cong U_1(Z\pi)/\pi \oplus SK_1(Z\pi)$ where $U_1(Z\pi)$ are the units of $Z\pi$ having augmentation 1. Then $\bar{\tau} \in U_1(Z\pi)/\pi$ is realized by a self-equivalence f: $X \to X$.*

PROOF. For any unit $u \in Z\pi$ of augmentation 1, let $u_y \in U_1Z(Z_n(y))$, $u_x \in U_1Z(Z_m(x))$ be defined by $u_yN_x = uN_x$, $u_xN_y = uN_y$. Note that u_x and u_y are also in $U_1Z\pi$. Also observe that $u_yN_y = \varepsilon(u_y)N_y = N_y$ and $u_xN_x = N_x$. Choose a unit $u \in U_1(Z\pi)$ representing $\bar{\tau}$ and let $v = u^{-1}$.

Let $\mathcal{C} = \mathcal{C}_*(\tilde{X})$ be given by (*) and $\mathcal{C}_v$ by the chain complex obtained from (*) by replacing the second boundary operator ∂_2 by

$$\partial_2^v = \begin{bmatrix} vN_x & 1-y & 0 \\ 0 & x-1 & N_y \end{bmatrix}.$$

Let $\mathcal{C}_v^+$ (similarly for $\mathcal{C}^+$) denote the expanded complex

$$\begin{array}{ccccccccc} (Z\pi)^3 & \overset{\partial_2^v}{\to} & (Z\pi)^2 & \overset{\partial_1}{\to} & Z\pi & \overset{\varepsilon}{\to} & Z & \to & 0 \\ \oplus & & \oplus & & & & & & \\ Z\pi & = & Z\pi & & & & & & \end{array}$$

Because $vN_x = v_yN_x$, $u_yN_y = N_y$ and $v_y^{-1} = u_y$, we have

$$\begin{bmatrix} N_x & 1-y & 0 & 0 \\ 0 & x-1 & N_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} v_y & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} v_yN_x & 1-y & 0 & 0 \\ 0 & x-1 & u_yN_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} vN_x & 1-y & 0 & 0 \\ 0 & x-1 & N_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The matrix

$$A = \begin{bmatrix} v_y & & & 0 \\ & 1 & & \\ & & u_y & \\ 0 & & & 1 \end{bmatrix}$$

is of the form

$$\begin{bmatrix} B & 0 \\ 0 & B^{-1} \end{bmatrix},$$

so it represents a simple basis change [C, p. 39]. Let $\bar{u}$: $C_2^+ \to (C_2^v)^+$ be given by the matrix

$$\begin{bmatrix} u & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}.$$

Then, by the above, $A \circ \bar{u}$: $C_2^+ \to C_2^+$ has the property that $\partial_2^+ A \circ \bar{u} = \partial_2^+$. One may now use the argument of **[DSI**, pp. 41–42], to alter the identity map $X^+ \to X^+$ ($X^+ = X \vee D^2$) to a map g: $X^+ \to X^+$ inducing $A \circ \bar{u}$ on C_2^+. This map is the identity on the one-skeleton of X^+ (and on C_0^+ and C_1^+). Because $A \circ \bar{u}$ is an isomorphism, $g \in \xi(X^+)$. Since g is the identity on the one-skeleton, one may compute $\tau(g) = \{A \circ \bar{u}\} = \{A\} + \{u\} = \bar{\tau}$, see Pl-P5 of **[W₂]**. By using the inclusion i: $X \to X^+$, its homotopy inverse $\bar{i}$ (both simple homotopy equivalences) and the fact that τ: $\xi(X^{(+)}) \to \mathrm{Wh}(\pi)$ is a crossed homormorphism, we have $\tau(f) = \bar{\tau}$, where $f = \bar{i} \circ g \circ i \in \xi(X)$. □

Clearly, if X is the realization of the presentation $\mathcal{P} = \{x_1, \ldots, x_n; x_1^{\tau_1}, \ldots, x_n^{\tau_n}, \{[x_i, x_j], 1 \leq i < j \leq n\}\}$ of the finite abelian group $G = Z_{\tau_1} \times \cdots \times Z_{\tau_n}$ $(\tau_i | \tau_{i+1}, i = 1, \ldots, n-1)$, then the argument above can be extended to show that

$$\xi X \twoheadrightarrow U_1(ZG)/G.$$

If $G = Z_p \times Z_{p^i}$, where p is prime, then $SK_1(ZG) = 0$ and homotopy and simple homotopy agree for $(G, 2)$-complexes. Because any two $(Z_m \times Z_n, 2)$-complexes at the same level have the same homotopy type **[D₃**, Corollary 1], we have the following corollary.

COROLLARY 2. *Suppose $SK_1(Z(Z_m \times Z_n)) = 0$. Two $(Z_m \times Z_n, 2)$-complexes have the same simple homotopy type if and only if they have the same Euler characteristic.* □

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