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ON INSERTION OF A CONTINUOUS FUNCTION

M. POWDERLY

ABSTRACT. Recently E. P. Lane proved that if space X has the weak C-insertion property and satisfies another condition, then X has the strong C-insertion property. This paper establishes the converse of this result.

In a recent paper, E. P. Lane [1] established two main results about the insertion of continuous functions. The second of these is:

THEOREM 3.1. *Let P_1 and P_2 be* C*-properties and consider the following condition*:

(α) *If g and f are functions on X such that $g \lessdot f$, g satisfies property P_1, and f satisfies property P_2, then there exists a sequence $\{A(f-g, 2^{-n})\}$ of lower cut sets for $f-g$ and there exists a sequence $\{F_n\}$ of subsets of X such that*

(i) $\{x/(f-g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n$, *and*

(ii) *for each n the sets $A(f-g, 2^{-n})$ and F_n are completely separated.*

If X satisfies the weak C*-insertion property for (P_1, P_2) and if X satisfies (α), then X satisfies the strong* C*-insertion property for (P_1, P_2). Conversely, if X satisfies the strong* C*-insertion property for (P_1, P_2) and $f-g$ satisfies the property P_1 (actually P_1 should have read P_2), then X satisfies (α).*

This paper shows that the last sentence of the above theorem can be strengthened to read:

Conversely, if X satisfies the strong C-insertion property for (P_1, P_2), then X satisfies (α). Thus (α) is a necessary and sufficient condition for a space with the weak C-insertion property to have the strong C-insertion property.

A property P defined relative to a real-valued function on a topological space is a C-property provided any constant function has property P and provided the sum of a function with property P and any continuous function also has property P. Let P_1 and P_2 be C-properties. A space X is said to have the weak C-insertion property for (P_1, P_2) iff for any functions g and f on X such that $g \lessdot f$, g has property P_1 and f has property P_2, then there exists a continuous function h on X such that $g \lessdot h \lessdot f$. A space X is said to have the strong C-insertion property for (P_1, P_2) iff for any functions g and f on X such that $g \lessdot f$, g satisfies P_1 and f satisfies P_2, then there exists a continuous function h on X such that $g \lessdot h \lessdot f$ and such that if $g(x) < f(x)$ for any x in X, then $g(x) < h(x) < f(x)$. If f is a real-valued function defined on a space X and if

$$\{x/f(x) < r\} \subset A(f, r) \subset \{x/f(x) \lessdot r\},$$

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for a real number r, then $A(f, r)$ is called a lower cut set.

We now prove the result stated above:

PROOF. Assume that X satisfies the strong C-insertion for (P_1, P_2). Let g and f be functions on X satisfying P_1 and P_2 respectively such that $g \lessdot f$. Thus there exists h continuous over X such that $g \lessdot h \lessdot f$ and such that if $g(x) < f(x)$ for any x in X, then $g(x) < h(x) < f(x)$. Now consider the functions 0 and $f - h$. 0 satisfies property P_1 and $f - h$ satisfies property P_2. Thus there exists function h_1 continuous over X such that $0 \lessdot h_1 \lessdot f - h$ and if $0 < (f - h)(x)$ for any x in X, then $0 < h_1(x) < (f - h)(x)$. We next show that

$$\{x/(f - g)(x) > 0\} = \{x/h_1(x) > 0\}.$$

If x is such that $(f - g)(x) > 0$, then $g(x) < f(x)$. Therefore $g(x) < h(x) < f(x)$. Thus $f(x) - h(x) > 0$ or $(f - h)(x) > 0$. Hence $h_1(x) > 0$. On the other hand, if $h_1(x) > 0$, then since $(f - h) \gtrdot h_1$ and $f - g \gtrdot f - h$, therefore $(f - g)(x) > 0$. For each n, let $A(f - g, 2^{-n}) = \{x/(f - g)(x) \lessdot 2^{-n}\}$,

$$F_n = \{x/h_1(x) \gtrdot 2^{-n+1}\},$$

and

$$k_n = \sup\{\inf\{h_1, 2^{-n+1}\}, 2^{-n}\} - 2^{-n}.$$

Since $\{x/(f - g)(x) > 0\} = \{x/h_1(x) > 0\}$, it follows that

$$\{x/(f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n.$$

We next show that k_n is a continuous function from X into $[0, 2^{-n}]$ which completely separates F_n and $A(f - g, 2^{-n})$. From its definition, it is clear that k_n is continuous over X. Let $x \in F_n$. Then, from the definition of k_n, $k_n(x) = 2^{-n}$. If $x \in A(f - g, 2^{-n})$, then since $h_1 \lessdot f - h \lessdot f - g$, $h_1(x) \lessdot 2^{-n}$. Thus $k_n(x) = 0$, according to the definition of k_n. Hence k_n completely separates F_n and $A(f - g, 2^{-n})$. This completes our proof.

A paper by Blatter and Seever [2] deals with closed lattice cones (of functions) on a set X, i.e., closed convex cones of bounded real-valued functions on X which contain the constants and which are closed under the lattice operation. The necessary and sufficient conditions are presented for the "interposition" of K between members of the pair (L, M), K, L, M being closed lattice cones. The results are given in terms of certain binary relations, called inclusions, on the power set of X. However, a lattice cone of functions on a set X and sets with C-insertion properties are in general not the same.

REFERENCES

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DEPARTMENT OF MATHEMATICS, WILLIAM PATERSON COLLEGE OF NEW JERSEY, WAYNE, NEW JERSEY 07470