PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 1, January 1981

SOME PROPERTIES OF H-SPACES OF RANK 2

C. A. McGIBBON[1]

ABSTRACT. We study H-spaces with 3 cells. After one suspension the attaching map for the top cell is shown to have order at most 2. For such H-spaces localized away from 2, we obtain a new proof of a classification theorem due to Zabrodsky. We also show that under suitable restrictions in the 3 cell case, the vanishing of all Whitehead products implies the existence of a multiplication.

1. Summary. Let X_l denote the localization [**8**], at a set of primes l, of a simply connected CW complex X of the form

$$X \simeq S^m \cup_a e^n \cup_b e^{m+n}.$$

Let α and β denote the corresponding attaching maps of X_l.

THEOREM 1. *If X_l is an H-space then the suspension $\Sigma\beta$ has order at most* 2. □

This upper bound is shown to be best possible by considering $X = SU(3)$ and letting l be the set of all primes.

An easy consequence of Theorem 1 is the following result due to Zabrodsky [**10**].

COROLLARY 2. *If X_l is an H-space and* 2 *is not in l, then the homotopy type of X_l is completely determined by*

(a) *the type* (m, n),

(b) *the set of primes l, and*

(c) *the homotopy class of the first attaching map α.* □

Given X_l as in Corollary 2, it follows from a classical result of James, [**3**], that X_l is a retract of $\Omega\Sigma X_l$. However with this hypothesis, more is true. Let C_α denote the mapping cone of α.

COROLLARY 3. *If X_l is as in Corollary* 2, *then X_l is a retract of $\Omega\Sigma C_\alpha$.* □

It is well known that Whitehead products (of all kinds) are trivial on an H-space. Does this property distinguish H-spaces among finite-dimensional spaces? G. J. Porter asked this question in [**6**]. Among the 3-cell complexes studied in this paper, the following is true.

Received by the editors December 28, 1979.

AMS (MOS) subject classifications (1970). Primary 55P45; Secondary 55Q15.

Key words and phrases. H-space, localization, Whitehead product.

[1]Supported in part by the NSF.

THEOREM 4. *Suppose X_l has the following properties:*
(a) *the first attaching map α is a suspension,*
(b) 2 *is not in l, and*
(c) *Whitehead products of all kinds are trivial on X_l.*
Then X_l is an H-space. □

2. Proofs. The proof of Theorem 1 amounts to producing a map

$$f: S^{m+n+1} \to \Sigma X_l$$

with degree 2 in H_{m+n+1}. To this end let $H(\mu)$: $\Sigma X \wedge X_l \to \Sigma X_l$ denote the Hopf construction, where μ is a multiplication on X_l. Let i: $S^m \to X_l$ and j: $C_\alpha \to X_l$ be the obvious embeddings.

Consider the composition

$$\Sigma S^m \wedge C_\alpha \xrightarrow{\Sigma i \wedge j + \Sigma j \wedge i} \Sigma X \wedge X_l \xrightarrow{H(\mu)} \Sigma X_l,$$

where the indicated sum is taken using the suspension comultiplication on the domain. It is straightforward to check that the first map, $\Sigma i \wedge j + \Sigma j \wedge i$, induces the trivial homomorphism in H_{2m+1} (since m is odd) and that the composite induces a map of degree 2 in H_{m+n+1}. By the Hurewicz theorem the composite map is null homotopic on the $2m + 1$ skeleton of the domain and hence factors through S_l^{m+n+1} to give the desired map f. ■

PROOF OF COROLLARY 2. Suppose that K and L are two H-spaces which satisfy the hypothesis of Corollary 2. There is a homotopy equivalence ϕ: $\Sigma K \to \Sigma L$ because of the equivalences $\Sigma K \simeq \Sigma C_\alpha \vee S_l^{m+n+1} \simeq \Sigma L$ implied by Theorem 1. If E: $K \to \Omega\Sigma K$ denotes the suspension map and r: $\Omega\Sigma L \to L$ is a retraction (which exists since L is a connected H-space), then the composition

$$K \xrightarrow{E} \Omega\Sigma K \xrightarrow{\Omega\phi} \Omega\Sigma L \xrightarrow{r} L$$

is easily seen to be a homotopy equivalence. ■

PROOF OF COROLLARY 3. Since X_l is an H-space, the inclusion j: $C_\alpha \to X_l$ can be extended to a map $\bar{j}$: $\Omega\Sigma C_\alpha \to X_l$. The adjoint of the retraction ρ: $\Sigma X_l \to \Sigma C_\alpha$ (given by Theorem 1), is clearly a right inverse for $\bar{j}$. ■

PROOF OF THEOREM 4. Since all spherical Whitehead products vanish on X_l it is easy to see that X_l has the rational homotopy type of $S^m \times S^n$ with m and n odd. Again, let i and j denote the natural inclusions of S^m and C_α, respectively, in X_l. Since C_α is a suspension, the generalized Whitehead product $[i, j]$ is defined [5]. By hypothesis, this product is trivial and thus there is a commutative diagram

$$\begin{array}{ccc} S^m \times C_\alpha & \xrightarrow{h} & X_l \\ \uparrow & \nearrow (i,j) & \\ S^m \vee C_\alpha & & \end{array}$$

Using this diagram one can verify that h induces an isomorphism in $H_{m+n}(\,;\mathbf{Z}_{(l)})$.

Since the product $[j, j]$ is also trivial there is a map g: $C_\alpha \times C_\alpha \to X_l$ which restricts to j on each factor. Take the Hopf construction $H(g)$, and consider the composition

$$\Sigma S^m \wedge C_\alpha \overset{\Sigma i \wedge 1 + \Sigma 1 \wedge i}{\to} \Sigma C_\alpha \wedge C_\alpha \overset{H(g)}{\to} \Sigma X_l.$$

It follows, just as in Theorem 1, that the top cell in ΣX_l is attached by a map of order at most 2. Since 2 is not in l, $\pi_* \Sigma C_\alpha$ has no 2-torsion and thus ΣC_α is a retract of ΣX_l. The adjoint of the retraction is a map ρ: $X_l \to \Omega\Sigma C_\alpha$.

A left inverse for ρ can be constructed using the James model $J(C_\alpha)$ for $\Omega\Sigma C_\alpha$ and the triviality of higher order Whitehead products on X_l. In more detail, the map g: $C_\alpha \times C_\alpha \to X_l$ factors through the second reduced product space $J_2(C_\alpha)$ [3] to give an extension

$$\begin{array}{ccc} J_2(C_\alpha) & & \\ \uparrow & \searrow j_2 & \\ C_\alpha & \overset{j}{\to} & X_l \end{array}$$

By Theorem 2.8 of [5], for each $n \geqslant 2$, the obstruction to extending a map j_n: $J_n(C_\alpha) \to X_l$ to $J_{n+1}(C_\alpha)$ is a certain higher order Whitehead product. Since this product is trivial by hypothesis, the extension exists. By induction on n, there is a map $\bar{j}$: $\varinjlim J_n(C_\alpha) = J(C_\alpha) \to X_l$ which extends the inclusion j: $C_\alpha \to X_l$. Since $J(C_\alpha) \simeq \Omega\Sigma C_\alpha$ [7] it follows that X_l is a retract of $\Omega\Sigma C_\alpha$ and hence that X_l is an H-space. ■

3. The case $X = SU(3)$. From the description of the attaching map for the top cell of ΣX given by James in [4, p. 128] it follows that the 9-cell in $\Sigma SU(3)$ is attached by the composition

$$S^8 \overset{\eta}{\to} S^7 \overset{\nu}{\to} S^4 \overset{i}{\to} S^4 \cup_\eta e^6.$$

To see that this is nontrivial it seems necessary to consider, in detail, the exact sequence

$$\to \pi_9\left(C_\eta, S^4\right) \overset{\partial}{\to} \pi_8 S^4 \overset{i}{\to} \pi_8 C_\eta \to .$$

By the Blakers-Massey Theorem [2] the relative group is generated by $[\iota_4, e_6]$ and $e_6 \circ \bar{\nu}_6$. Applying the boundary operator

$$\partial[\iota, e_6] = [\iota, \iota \circ \eta_4] = E\nu' \circ \eta_7 \quad \text{by } [\mathbf{1}],$$

and

$$\partial(e_6 \circ \bar{\nu}_6) = \eta_4 \circ \nu_5 = E\nu' \circ \eta_7 \quad \text{by } [\mathbf{9}, \text{p. } 44].$$

since

$$\pi_8 S^4 = \{\nu_4 \circ \eta_7\} \oplus \{E\nu' \circ \eta_7\} \quad \text{by } [\mathbf{9}, \text{p. } 43],$$

it follows by exactness that the attaching map $i_*(\nu_4 \circ \eta_7)$ is nontrivial as claimed. ■

References

1. M. G. Barratt, I. M. James and N. Stein, *Whitehead products and projective spaces*, J. Math. Mech. **9** (1960), 813–819.

2. A. L. Blakers and W. S. Massey, *Products in homotopy theory*, Ann. of Math. (2) **58** (1953), 295–324.

3. I. M. James, *Reduced product spaces*, Ann. of Math. (2) **62** (1955), 170–197.

4. ______, *On sphere-bundles over spheres*, Comment Math. Helv. **35** (1961), 126–135.

5. G. J. Porter, *Higher order Whitehead products*, Topology **3** (1965), 123–135.

6. ______, *Spaces with vanishing Whitehead products*, Quart. J. Math. Oxford Ser. **16** (1965), 77–85.

7. D. Puppe, *Some well known weak homotopy equivalences are genuine homotopy equivalences*, Symposia Mathematica, Vol. V (INDAM, Rome, 1969/1970), Academic Press, London, 1971, pp. 363–374.

8. D. Sullivan, *Geometric topology*, Part I, M.I.T. Notes, 1970.

9. H. Toda, *Composition methods in homotopy groups of spheres*, Ann. of Math. Studies, No. 49, Princeton Univ. Press, Princeton, N.J., 1962.

10. A. Zabrodsky, *On rank* 2 mod *odd H-spaces*, New Developments in Topology, London Math. Soc. Lecture Notes Series, no. 11, Cambridge Univ. Press, London, 1974, pp. 119–128.

Department of Mathematics, University of Pennsylvania, Philadelphia, Pennsylvania 19104

Current address: Department of Mathematics, Indiana University, Bloomington, Indiana 47401