PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 1, January 1981

ON THE TWO-REALIZABILITY OF CHAIN COMPLEXES

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ABSTRACT. We give a sufficient condition which insures the realizability of a two-dimensional chain complex satisfying Wall's condition by a two-dimensional CW-complex.

Let π be a group generated by $x_1, \ldots, x_n$. Let C_* be a two-dimensional chain complex

$$\begin{array}{cccccc} C_*: & C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 \\ & \| & & \| & & \| \\ & Z\pi^m & & Z\pi^n & & Z\pi \end{array}$$

satisfying these conditions: (i) $H_1(C_*) = 0$, (ii) $H_0(C_*) \cong Z$, and (iii) the boundary operator $\partial_1 = (x_1 - 1, \ldots, x_n - 1)$. In [7], Wall conjectured that under the above conditions C_* could be realized as the homology chains of a two-dimensional CW-complex. However, in [3], Dunwoody gave an example of a chain complex C_* which satisfied Wall's conditions but was not realizable. The purpose of this note is to give a sufficient condition which insures the realizability of C_* by a two-dimensional CW-complex.

Suppose there is a presentation $\mathcal{P} = (x_1, \ldots, x_n: R_1, \ldots, R_m)$ for the group π (see the Remark 2 below). We let $F = F(x_1, \ldots, x_n)$, the free group generated by $x_1, \ldots, x_n$ and $R = N_F\{R_1, \ldots, R_m\}$, the normal closure in F of $R_1, \ldots, R_m$. Let $\bar{F} = F(r_1, \ldots, r_m)$, the free group on symbols $r_1, \ldots, r_m$, and let $\psi: F * \bar{F} \to F$ be the homomorphism taking $x_i \mapsto x_i$ and $r_j \mapsto R_j$.

Now if $\varphi: F \to F/R = \pi$ is the natural projection and $\kappa: F \to C_1$ is the crossed homomorphism, then by Corollary 4.4, p. 655 of [8], we can find words $W_1, \ldots, W_m$ such that the matrix $\|\kappa(W_i)\|^\varphi$ is the boundary operator ∂_2. The problem is that $N_F\{W_1, \ldots, W_m\}$ may not generate the entire normal subgroup R. Because each $W_i \in R$, we can write

$$W_i = \prod_{k=1}^{m_i} (x_{ik} R_{ik} x_{ik}^{-1})^{\varepsilon_{ik}}$$

where $R_{ik} = R_j$, for some j, $1 < j < m$, $x_{ik} \in F$, $\varepsilon_{ik} = \pm 1$. Let $w_i = \prod_{k=1}^{m_i} (x_{ik} r_{ik} x_{ik}^{-1})^{\varepsilon_{ik}}$ where if $R_{ik} = R_j$ in W_i, then $r_{ik} = r_j$ in w_i. Let J denote the

Received by the editors May 30, 1979 and, in revised form, August 17, 1979.
AMS (MOS) subject classifications (1970). Primary 55A05, 20F05.

0002-9939/81/0000-0029/$02.00

$m \times m$ matrix

$$J = \begin{Vmatrix} \frac{\partial w_1}{\partial r_1} & \cdots & \frac{\partial w_1}{\partial r_m} \\ \vdots & & \\ \frac{\partial w_m}{\partial r_1} & \cdots & \frac{\partial w_m}{\partial r_m} \end{Vmatrix}.$$

Now we prove

LEMMA. *The following are equivalent.*

(1) $N_{F * \bar{F}}\{w_1, \ldots, w_m\} = N_{F * \bar{F}}\{r_1, \ldots, r_m\}$.

(2) $\{x_1, \ldots, x_n, w_1, \ldots, w_m\}$ *forms a generating set for* $F * \bar{F}$.

(3) J *has a right inverse.*

PROOF. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3). Suppose $\{x_1, \ldots, x_n, w_1, \ldots, w_m\}$ forms a generating set for $F * \bar{F}$. Then by the Inverse Function Theorem [**1**], the Jacobian which has the form

$$\begin{Vmatrix} I_n & 0 \\ A & J \end{Vmatrix}$$

has a right inverse B. By a result of Kaplansky [**6**], B is a two-sided inverse so that J has a right inverse.

(3) $\Rightarrow$ (1). Suppose J has a right inverse H. Then the Jacobian which has the form

$$\begin{Vmatrix} I_n & 0_m \\ A_{m \times n} & J_m \end{Vmatrix}$$

has a right inverse

$$\begin{Vmatrix} I_n & 0_m \\ -H_m A_{m \times n} & H_m \end{Vmatrix}.$$

Thus, by the Inverse Function Theorem $\{x_1, \ldots, x_n, w_1, \ldots, w_m\}$ is a generating set for $F * \bar{F}$. We claim $N_{F * \bar{F}}\{w_1, \ldots, w_m\} \overset{\text{def}}{=} \bar{N} = N \overset{\text{def}}{=} N_{F * \bar{F}}\{r_1, \ldots, r_m\}$. Clearly $\bar{N} \subseteq N$. Therefore we have the short exact sequence $1 \to N/\bar{N} \to F * \bar{F}/\bar{N} \to F * \bar{F}/N \to 1$. Since $F * \bar{F}/\bar{N} \cong F$ and $F * \bar{F}/N \cong F$, we must have $N = \bar{N}$. This completes the proof of the lemma.

REMARK 1. Let J^α denote the image of J under the abelianizing homomorphism α acting on $Z(F * \bar{F})$. Then $N_{F * \bar{F}}\{w_1, \ldots, w_m\} = N_{F * \bar{F}}\{r_1, \ldots, r_m\}$ only if the determinant $\det J^\alpha$ is a unit in $Z(F * \bar{F})^\alpha$. For $N_{F * \bar{F}}\{w_1, \ldots, w_m\} = N_{F * \bar{F}}\{r_1, \ldots, r_m\}$ implies that $\{x_1, \ldots, x_n, w_1, \ldots, w_m\}$ is a basis for $F * \bar{F}$. By Corollary 2 of [**1**], $\det \bar{J}$ is a unit in $Z(F * \bar{F})^\alpha$ where

$$\bar{J} = \begin{Vmatrix} I_n & 0 \\ A & J \end{Vmatrix};$$

therefore $\det \bar{J} = \det J$ is a unit in $Z(F * \bar{F})^\alpha$.

THEOREM. *Notation as above. The chain complex C_* is realizable by a two-dimensional CW-complex if the Jacobian J has a right inverse.*

PROOF. By the above lemma, we have $N_{F*\bar{F}}\{w_1, \ldots, w_m\} = N_{F*\bar{F}}\{r_1, \ldots, r_m\}$. Applying ψ we get $N_F\{W_1, \ldots, W_m\} = R$. Therefore if R denotes the cellular model of the presentation $\mathfrak{R} = (x_1, \ldots, x_n: W_1, \ldots, W_m)$ for π, then the universal cover $\tilde{R}$ of R realizes the given chain complex C_*.

REMARK 2. It is possible that there does not exist a presentation $\mathfrak{P}$ with n generators and m relators. This would happen if the relation module $\bar{R} = \ker \partial_1$ is generated by fewer than m elements. (See Dyer [4].)

REMARK 3. Because both

$$\{x_1, \ldots, x_n, w_1, \ldots, w_m\} \quad \text{and} \quad \{x_1, \ldots, x_n, r_1, \ldots, r_m\}$$

form generating sets for the free group $F * \bar{F}$, we can convert one set to the other using Nielsen transformations (see [5]). This implies that P and R which are the cellular models of the presentations $\mathfrak{P}$ and $\mathfrak{R}$, respectively, have the same simple homotopy type.

EXAMPLE. Let π be the group $Z_5 \times Z_5 \times Z_5$ generated by a, b, and c, and let C_* be the chain complex

$$\begin{array}{ccccc} C_2 & \overset{\partial_2}{\rightarrow} & C_1 & \overset{\partial_1}{\rightarrow} & C_0 \\ \| & & \| & & \| \\ Z\pi^6 & & Z\pi^3 & & Z\pi \end{array}$$

where $\partial_1 = (a-1, b-1, c-1)$ and ∂_2 is the matrix

$$\left\| \begin{array}{ccc} \dfrac{a^5-1}{a-1} & 0 & 0 \\ 0 & \dfrac{b^5-1}{b-1} & 0 \\ 0 & 0 & \dfrac{c^5-1}{c-1} \\ 1-b & a-1 & 0 \\ 1-c & 0 & a-1 \\ 0 & 1-c & b-1 \end{array} \right\|.$$

Let $\mathfrak{P}$ be the presentation $(a, b, c: a^5, b^5, c^5, [a^4, b], [a, c], [b, c])$. We see that we can take $W_1 = a^5$, $W_2 = b^5$, $W_3 = c^5$, $W_4 = aba^{-5}b^{-1}[a^4, b]^{-1}a^5a^{-1}$, $W_5 = [a, c]$, and $W_6 = [b, c]$ and the corresponding Jacobian J has a right inverse. Therefore the chain complex C_* is realizable. Indeed,

$$\mathfrak{R} = \left(a, b, c: a^5, b^5, c^5, [a, b], [a, c], [b, c]\right),$$

and the cellular models P and R have the same simple homotopy type.

References

1. J. S. Birman, *An inverse function theorem for free groups*, Proc. Amer. Math. Soc. **41** (1973), 634–638.

2. W. Cockcroft and R. M. F. Moss, *On the two-dimensional realizability of chain complexes*, J. London Math. Soc. **11** (1975), 257–262.

3. M. J. Dunwoody, *Relation modules*, Bull. London Math. Soc. **4** (1972), 151–155.

4. M. N. Dyer, *Spaces dominated by finite two-dimensional CW-complexes* (to appear).

5. W. Magnus, A. Karrass and D. Solitar, *Combinatorial group theory*, Pure and Appl. Math., vol. 13, Interscience, New York, 1966.

6. M. S. Montgomery, *Left and right inverses in group algebras*, Bull. Amer. Math. Soc. **75** (1969), 539–540.

7. C. T. C. Wall, *Finiteness conditions for CW-complexes*. II, Proc. Roy. Soc. Ser. A **295** (1966), 129–139.

8. R. C. Lyndon, *Cohomology theory of groups with a single defining relation*, Ann. of Math. (2) **52** (1950), 650–665.

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