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σ-COHERENT CONTINUA ARE HEREDITARILY LOCALLY CONNECTED

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ABSTRACT. A σ-coherent continuum is one in which every descending sequence of connected sets has a connected intersection. In this paper it is proved that such continua are hereditarily locally connnected. An example is given to show that the converse is not true.

1. Introduction. A topological space is called σ-coherent if the intersection of evey descending sequence of connected sets is connected. This concept was introduced in [1] and used as a condition on the domain of a real-valued function to obtain monotonicity. In [2] the concept was used in the following theorem. If f: $X \to Y$ is a monotone connected function from the locally connected metric continuum X onto the σ-coherent metric continuum Y, then f is continuous. In the same paper it was conjectured that if X and Y are hereditarily locally connected metric continua, then such a function f must be continuous. This is still unresolved and it is natural to investigate the relationship between σ-coherence and hereditary local connectivity for metric continua. In [2] an example was given to show that an hereditarily locally connected metric continuum need not be σ-coherent and Example 1 below shows that even a regular continuum may fail to be σ-coherent. It is an open question whether a σ-coherent continuum must be regular. In this paper it is shown that a σ-coherent continuum must be hereditarily locally connected.

The reader is referred to [4] and [5] for definition of terms not given here.

2. Results. The following lemmas establish that a σ-coherent metric continuum is hereditarily arcwise connected, a result which is used in the proof of the main theorem.

LEMMA 1. *If X is a compact σ-coherent metric continuum, then X is hereditarily decomposable.*

The proof, being straightforward, is omitted.

LEMMA 2. *If X is a compact σ-coherent metric continuum, then X is hereditarily arcwise connected.*

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PROOF. let p and q be distinct points of X and M an irreducible subcontinuum joining p and q. Since M is hereditarily decomposable, Theorem 2.2 of [3] implies there is a monotone continuous function g: $M \to [0, 1]$ such that $g(p) = 0$ and $g(p) = 1$. It will be shown that g is one-to-one and hence M is an arc. Let x be in (0, 1) and assume a and b are distinct points of $g^{-1}(x)$. Choose a sequence $\{x_n\}_{n=1}^{\infty}$ in (0, 1) such that for each n, $x_n < x_{n+1} < x$, and such that limit $x_n = x$. If $C_n = g^{-1}([x_n, x)) \cup \{a, b\}$ for $n \geq 1$, then $\{C_n\}_{n=1}^{\infty}$ is a descending sequence of connected sets such that $\bigcap_{n=1}^{\infty} C_n = \{a, b\}$ which is not connected. This contradicts X being σ-coherent and hence $g^{-1}(x)$ must be a singleton set. A similar result is obtained if $x = 0$ or $x = 1$. Thus g is one-to-one and M is an arc.

Since σ-coherence is an hereditary property, the conclusion of the lemma follows.

The following result is the principal theorem of the paper.

THEOREM. *If X is a compact, σ-coherent continuum, then X is locally connected.*

Actually, the following slightly stronger theorem is proved from which the principal theorem stated above follows.

THEOREM. *If X is a compact, σ-coherent continuum, then X contains no sequence of mutually disjoint continua $\{M_i\}_{i=1}^{\infty}$ converging to a nondegenerate continuum M such that $M - \bigcup_{i>0} M_i$ is uncountable.*

PROOF. Assume the contrary and suppose X contains a sequence $\{M_i\}_{i=1}^{\infty}$ of mutually disjoint continua converging to a nondegenerate continuum M such that $M - \bigcup_{i>0} M_i$ is uncountable. In this argument an arc may be a singleton set and by an arc from a set H to a set K is meant an ordered arc with first point in H, last point in K, no interior point in $H \cup K$ and which is degenerate if it contains a point of $H \cap K$. By Lemma 2, X is arcwise connected so there is a sequence $\{\beta_i\}_{i=1}^{\infty}$ such that for each positive integer n, β_n is an arc from M_n to M. Define inductively a sequence of arcs such that $a_1 = \beta_1$ and for each integer $n > 1$, a_n is a subarc of β_n from M_n to $M \cup \bigcup_{i=1}^{n-1} a_i$. Note that if i and j are positive integers, $i \neq j$, then $a_i \neq a_j$. Let $A = \bigcup_{i>0}\{a_i\}$. We next define a finite sequence of subcollections of A with union A. Let $A_1^1 = \{a_1\}$ and let A_1^2 denote the set of all arcs in $A - A_1^1$ with last point in a_1. Continue to define a sequence $\{A_1^i\}_{i=1}^{\infty}$ of subcollections of A such that for each positive integer $n > 1$, A_1^n is the set of all arcs in $A - \bigcup_{i=1}^{n-1} A_1^i$ with last point in a member of A_1^{n-1}. Let $A_1 = \bigcup_{i>0} A_1^i$. If $A = A_1$, then the process terminates. If not, then let n_1 denote the least positive integer j such that a_j is in $A - A_1$, let $A_2^1 = \{a_{n_1}\}$, and, as before, define a sequence $\{A_2^i\}_{i=1}^{\infty}$ such that for each positive integer $n > 1$, A_2^n is the set of all arcs in $A - \bigcup_{i=1}^{n-1} A_2^i$ with last point in some member of A_2^{n-1}. Next define $A_2 = \bigcup_{i>0} A_2^i$ and if $A = A_1 \cup A_2$, then our process is complete. Otherwise we define n_2 as the least positive integer j such that a_{n_2} is in $A - (A_1 \cup A_2)$ and in a similar manner define A_3. We next show that there is a positive integer N such that $A = \bigcup_{i=1}^{N} A_i$ so the process must terminate. If the process does not terminate then there is determined an infinite sequence $\{a_{n_i}\}_{i=1}^{\infty}$ of mutually disjoint arcs, each having its last point in M. Since $M - \bigcup_{i>0} M_i$ is uncountable and for each positive integer i, $a_{n_i} \cap M$ is degenerate, then there are two points x and y in $M - \bigcup_{i>0}(M_{n_i} \cup a_{n_i})$. For each positive integer k, let $L_k = \bigcup_{i=k}^{\infty}(M_{n_i} \cup \alpha_{n_i}) \cup \{x\} \cup \{y\}$ and note that L_k is connected. But then the sequence $\{L_i\}_{i=1}^{\infty}$ is a nested sequence of connected sets whose intersection is $\{x, y\}$ which is not connected, contrary to the assumption that X is σ-coherent. It follows that there is a least integer N such that $A = \bigcup_{i=1}^{N} A_i$.

Next we show that if each of u and v is a positive integer and $1 \leqslant u \leqslant N$, then A_u^v is finite. If not, then there is an integer u with $1 \leqslant u \leqslant N$ and a positive integer v such that A_u^v is finite and A_u^{v+1} is infinite. Thus there is an arc β in A_u^v and an infinite sequence $\{a_{m_i}\}_{i=1}^{\infty}$ of arcs in A_u^{v+1} each having its last point in β. By construction, β has at most one point in common with M so there is a point x in $M - \beta$. For each positive integer k, let $L_k = \bigcup_{i>k}(\alpha_{m_i} \cup M_{m_i}) \cup \beta \cup \{x\}$ and note that $\{L_i\}_{i=1}^{\infty}$ is a nested sequence of connected sets whose intersection is $\beta \cup \{x\}$ which is not connected, contrary to hypothesis. Since the arcs of the sequence $\{a_i\}_{i=1}^{\infty}$ are distinct, it follows that there is an integer u, $1 \leqslant u \leqslant N$, such that for each positive integer v, A_u^v is nonempty.

We next show there is a sequence $\{\gamma_i\}_{i=1}^{\infty}$ such that for each positive integer n, γ_{n+1} has its last point in γ_n and γ_n is in A_u^n. Let γ_1 be the member of A_u^1. If n is a positive integer and δ_1 is an arc in A_u, then by an n-chain from δ_1 is meant a finite sequence of arcs, $\delta_1\delta_2, \ldots, \delta_n$, such that (1) for each integer i with $1 < i \leqslant n$, δ_i has its last point in δ_{i-1}, and (2) there is an integer $m > 0$ such that for each integer i with $1 \leqslant i \leqslant n$, δ_1 is in A_u^{m+i}. There is an arc δ in A_u^2 such that for each positive integer n, there is an n-chain from δ. If not then for each member δ of A_u^2 there is an integer n_δ such that there is no n_δ-chain from δ. But then if $N = \max\{n_\delta | \delta \in A_u^2\}$ then A_u^{N+1} is vacuous which is a contradiction. Thus there is an arc γ_2 in A_u^2 such that for each positive integer n, there is an n-chain from γ_2. Similarly there is an arc γ_3 in A_u^3 with last point in γ_2 and such that for each positive integer n, there is an n-chain from γ_3. This process may be continued to determine an infinite sequence $\{\gamma_i\}_{i=1}^{\infty}$ of arcs such that for each positive integer n, $\gamma_n \in A_u^n$ and the last point of γ_{n+1} is in γ_n. Note that if each of i and j is a positive integer and $|i - j| > 1$, then $\gamma_i \cap \gamma_j$ is vacuous. Let x and y denote distinct points of $M - \gamma_1$, and for each positive integer i, let k_i denote the integer such that $a_{k_i} = \gamma_i$. For each positive integer n, let $L_n = \bigcup_{i>n}(a_{k_i} \cup M_{k_i}) \cup \{x\} \cup \{y\}$. Once again we have a nested sequence $\{L_i\}_{i=1}^{\infty}$ of connected sets whose intersection is not connected, contrary to hypothesis, and our proof is complete.

3. Examples and remarks.

EXAMPLE 1. This example shows that a regular continuum may fail to be σ-coherent. Let $A = [0, 1]$ and let D be the set of dyadic rationals between 0 and 1. For each x in D of the form $(2p + 1)/2^n$, where $n \geqslant 1$ and $0 \leqslant p \leqslant 2^{n-1} - 1$, let C_x be the semicircle in the upper half-plane with center at x and radius $1/2^n$. Then $X = A \cup (\bigcup_{x \in D} C_x)$, with the relative topology from the plane, is a regular continuum which is not σ-coherent.

This example suggests that a σ-coherent continuum cannot contain a continuum of condensation. However, the example on p. 247 of [3], which is an example of a continuum of condensation which is not a convergence continuum, shows that this is false. The following example shows that even a continuum with no continuum of condensation (and hence regular) may fail to be σ-coherent.

EXAMPLE 2. Let $A = [0, 1]$ and, for $n \geq 1$, let C_n be the semicircle in the upper half-plane with center at $1/2$ and radius $1/2^n$. Let $X = A \cup (\bigcup_{n \geq 1} C_n)$. For each n let $K_n = (\bigcup_{k \geq n} C_k) \cup (A - \{1/2\})$. Then for each n, K_n is connected, $K_{n+1} \subset K_n$, and $\bigcap_{n=1}^{\infty} K_n = A - \{1/2\}$, which is not connected. Thus, X is not σ-coherent.

The property of being acyclic (in the sense of containing no simple closed curve) does not imply σ-coherence. However, the fact that σ-coherence is equivalent to local connectivity in the acyclic case follows from the observation that every dendrite is σ-coherent.

Finally, it is simple to show that σ-coherence is preserved under monotone maps. It would be interesting to know if this is true for confluent maps.

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