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A REMARKABLE SIMPLE CLOSED CURVE: REVISITED

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ABSTRACT. It is shown that the pathology of R. H. Fox's remarkable simple closed curve is in a sense explained below more complicated than that of some examples of the well-known Fox-Artin paper.

A classical example of Fox-Artin shows that the union of two tamely embedded arcs whose intersection is a common end-point may be wildly embedded [3, Example 1.4]. Arcs formed in this way are called mildly wild. A classification theorem exists for such sets if the union is locally peripherally unknotted, LPU, also called Wilder arcs [4]. While the concepts of LPU and LU [7] have served to classify the local pathology of the examples of [3], a more delicate invariant seems necessary to relate the embedding of [5] to those in [3]. There is the possibility that an appropriately chosen arc from "The remarkable simple closed curve" might be "less wildly" embedded than the arcs in [3]. This note dispells such feasability. To make this more precise a new embedding condition is introduced related to LPU. First, however, we review Fox's concept of almost unknotted.

A simple closed curve Γ in three-space $\mathfrak{R}^3$ is called almost unknotted if there is a point p and a neighborhood U of p such that for any neighborhood V of p there is a homeomorphism ϕ of $\mathfrak{R}^3$ on $\mathfrak{R}^3$ such that

(i) ϕ is the identity on V,

(ii) $\phi|\Gamma \setminus U$ is a subset of a plane.

It is clear that if q is another point and Γ has the same property at q, then Γ is unknotted.

There is a local property suggested by the above. An arc X is called locally almost unknotted at p if for some neighborhood U of p, no matter how small a neighborhood V of p is chosen, there is a neighborhood W of p and a homeomorphism ϕ of $\mathfrak{R}^3$ on $\mathfrak{R}^3$ such that

(i) $\phi =$ identity on W and

(ii) $\phi|U \setminus V$ is a subset of a plane. We abbreviate this property by LAU (locally almost unknotted).

Before introducing the next definition recall the definition of local peripheral unknottedness for a 1-manifold in $\mathfrak{R}^3$ [7]. Let p be an interior point (boundary point) of X and $\varepsilon > 0$. It is required that there be a topological 2-sphere K whose interior contains p such that

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(i) diam $K < \varepsilon$,

(ii) card $K \cap X = 2$ or 1 according as p is an interior or boundary point of X. See [3, Examples 1.1 or 1.2] for the failure of this property.

A 1-manifold $X \subset \mathfrak{R}^3$ is called weakly peripherally unknotted at p if for each $\varepsilon > 0$ there is a homeomorphism h_ε of $\mathfrak{R}^3$ on $\mathfrak{R}^3$ such that

(0) $h_\varepsilon(p) = p$,

(i) diam $h_\varepsilon^{-1}(K) < \varepsilon$,

(ii) card $K \cap X = 2$ or 1 according as p is an interior or boundary point of X. We abbreviate this property by WPU. Note "The remarkable simple closed curve" has this property at each point.

If one may take $h_\varepsilon =$ identity for all $\varepsilon > 0$, this becomes LPU.

As a preliminary to the main result (Theorem 2) we have the following.

Theorem 1. *Let X be an arc that is locally tame modulo p, p an end-point of X. Then X is LAU at p.*

Proof. If the penetration index of X at p is 1, a sequence of space homeomorphisms can be defined carrying X onto an interval. Hence X is tame. In general, by the smoothing techniques of [1] or [9], $X \setminus p$ may be taken as locally polyhedral. If the other end-point of X is q, let X be ordered from q to p. Also, let $F(p, \varepsilon)$ denote the surface of a sphere of radius ε, centered at p. Let Y_ε denote the subarc of X from q to t_ε, the last point on X in the assigned order. Then Y_ε is a finite polygonal arc and by elementary means (see Graueb [6]) can be straightened out to a segment Y_ε' leaving a neighborhood of p pointwise fixed. Denote this semilinear homeomorphism by f_ε. Let a neighborhood of p that is pointwise fixed by f_ε be Z_ε. We note X becomes $f_\varepsilon(X)$ and the part of $f_\varepsilon(X)$ exterior to $F(p, \varepsilon)$ is a segment and hence lies in a plane. If $\varepsilon' > \varepsilon$, $F(p, \varepsilon')$ meets $f_\varepsilon(X)$ in at most one point. Then $K = f_\varepsilon^{-1}[F(p, \varepsilon')]$ is a topological 2-sphere that meets X in a single point if $\varepsilon' - \varepsilon$ is sufficiently small. Since the points of Z_ε have remained pointwise fixed, X is LAU at p.

Remark 1. The segment Y_ε' referred to above may be taken to lie on a line through p.

Remark 2. Among the arcs locally tame modulo an end-point, the LAU condition is no further restriction of the embedding.

Remark 3. Let X be a union of two arcs X' and X'' meeting at p. Assume $X' \setminus p$ and $X'' \setminus p$ are locally tame. If X has penetration index $= 2$ at p, then X' and X'' are each tame and X is at most mildly wild [8]. Applying the above calculations to X' and X'', we see that the hypothesis that X be LAU at p implies that the homeomorphisms used in straightening X' and X'' be consistent, i.e. there is a single homeomorphism that straightens each of X', X'' except for a neighborhood of p. This will be taken as our definition of LAU at an interior point of an arc below.

Theorem 2. *Among the mildly wild arcs, LAU and WPU are equivalent properties.*

Proof. Let $X = X' \cup X''$ be mildly wild, where $X' \cap X'' = \{p\}$, X', X'' are tame and X is LAU at p. Given $\varepsilon > 0$, define $U = S(p, \varepsilon) \cap X$. Let V be the component of $S(p, \varepsilon/3) \cap X$ determined by p. By the LAU property, there is a neighborhood W of p and a homeomorphism ξ of $\mathbf{R}^3$ on $\mathbf{R}^3$ such that

(i) $\xi | W =$ identity,

(ii) $\xi(U \setminus V)$ is a subset of plane π. Diminishing W if necessary we assume $W = S(p, \delta)$ and for all x in W the arc xp has a diameter $< \varepsilon/3$. If q_{-1} and q_{+1} are the end-points of X, let X be reparametrized so that q_{-1} corresponds to $t = -1$, p to $t = 0$ and $t = +1$ to q_1. Let A denote the first component of $(q_{-1}p) \cap \pi$ with one boundary component on $F(p, \varepsilon')$, $\varepsilon' > \varepsilon$, and one on $F(p, \varepsilon)$. Let B denote the last component of $(pq_1) \cap \pi$ with one boundary component on $F(p, \varepsilon)$ and one on $F(p, \varepsilon')$. All other components of $\pi \cap (X \setminus W)$ can be pushed into $S(p, \varepsilon) \setminus W$ without moving the points of W by a homeomorphism ζ_2. The set $\{S(p, \varepsilon') \setminus S(p, \varepsilon)\} \cap \pi$ is an annulus with two components of

$$\{\{S(p, \varepsilon') \setminus S(p, \varepsilon)\}\} \cap X,$$

call them A and B, stretching from $F(p, \varepsilon')$ to $F(p, \varepsilon)$. Let H be a simple closed curve in this annulus piercing both of A and B just once. Then define $K = H \times [-\varepsilon, +\varepsilon] \cup (\text{Int } H \times \varepsilon) \cup \{(\text{Int } H) \times -\varepsilon\}$. Clearly K is a topological 2-sphere containing p in its interior and $K \cap X =$ a pair of points, one on A, one on B and diam $K < 2\varepsilon' + 2\varepsilon \leqslant 4\varepsilon'$ (using a rectangular metric).

By construction, diam $K < 3\varepsilon$ if $\varepsilon' - \varepsilon$ is sufficiently small. Taking $h = (\zeta_2\zeta)^{-1}$ we have

(0) $h(p) = p$,

(i) diam $K < 3\varepsilon$,

(ii) card $K \cap h^{-1}(X) = 2$, i.e., X is WPU at p.

In the converse direction, suppose $X = A \cup B$, where A and B are tame and $A \cap B = \{p\}$ is a common end-point. Assuming X is WPU at p we want to prove X is LAU at p. Given $\varepsilon > 0$, there is a topological 2-sphere K containing p in its interior and a homeomorphism h of $\mathbf{R}^3$ on $\mathbf{R}^3$ such that

(0) $h(p) = p$,

(i) diam $h^{-1}(K) < \varepsilon$,

(ii) $K \cap X = a \cup b$, $a \in A$ and $b \in B$.

Let A_1 denote the component of $A' \setminus a$ not containing p and B_1 the component of $B' \setminus b$ not containing p. Then A_1 and B_1 are tame disjoint arcs defined by the WPU property. By the smoothing techniques of [1] or [9] we may choose $\overline{A}_1$ and $\overline{B}_1$ as polygonal arcs. Again, by elementary techniques $\overline{A}_1 \cup \overline{B}_1$ may be flattened out into a plane leaving a neighborhood S of p pointwise fixed by a homeomorphism g.

If U is any neighborhood of p large enough to contain $X \setminus g(A_1 \cup B_1)$, then given any $V(p) \subset U$ there is a $W(p) \subset V \subset S$ so that

(i) $g | W =$ identity,

(ii) $g | U \setminus V$ is a subset of a plane.

This serves in the definition of LAU for X at p.

THEOREM 3. *Let* Γ *again denote "The remarkable simple closed curve." Then* Γ *fails to be either locally unknotted* (*LU*) *or locally peripherally unknotted* (*LPU*) *at* p.

PROOF. Suppose Γ is LU at p. Then there is a disk D containing a neighborhood of p in Γ. Since Γ is locally tame mod p, there is no loss in choosing D locally polyhedral mod p. Let $Y = A \cup B$ be a neighborhood of p in Γ where A is a straight line interval with p as an end-point and B the closure of the complement in Y. Then A and B are equivalently embedded in $\mathbf{R}^3$ by Theorem 5 of **[2]**. Hence A and B are both tame. The existence of D means $A \cup B$ has the A1P at p. Thus $A \cup B$ is tame by **[8]**, a contradiction. Hence Γ is locally knotted at p (i.e. no such D exists).

If Γ were LPU at p, Γ would be expressible as a union of two nonoverlapping arcs qap, qbp denoted by A, B, respectively. At least one of A, B is tame, say A. There is a disk D whose boundary contains A and D may be chosen tame. By choosing D carefully we can arrange that $D \cap B = \{p\}$. Since Γ is LPU at p, B is LPU at p. Hence B is tame. It follows that $A \cup B$ is part of a tame arc and hence tame, a contradiction.

Thus, the embedding of Γ at p is more complicated than that of either 1.2 or 1.4 of **[3]**. Γ is not a union of two tame nonoverlapping arcs but is a countable union of such tame nonoverlapping arcs.

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