SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

A SYMMETRIC SPACE OF NONCOMPACT TYPE HAS NO EQUIVARIANT ISOMETRIC IMMERSIONS INTO THE EUCLIDEAN SPACE

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Let $M$ be a Riemannian globally symmetric space of noncompact type. Let $G$ be the identity-connected component of the isometry group of $M$. Then

2. $G$ is a semisimple Lie group, having noncompact simple factors. For a proof cf. [2, p. 194].

**Lemma.** Let $G$ be a semisimple Lie group, having no compact simple factors. Then every differentiable morphism from $G$ into the group $\text{In}$ of isometries of $\mathbb{R}^n$, is the trivial morphism.

**Proof.** Replacing $G$ by a finite covering we may and will assume that $G$ is a direct product of simple Lie groups, none of which is compact. The group $\text{In}$ of isometries of the Euclidean $n$-space is known to be the semidirect product of the orthogonal group and $\mathbb{R}^n$, $\mathbb{R}^n$ being the normal subgroup. Therefore the projection of $\text{In}$ onto the orthogonal group is a differentiable group morphism.

Let $f$ be a differentiable morphism from $G$ into $\text{In}$. Then $f$ followed by the projection of $\text{In}$ onto the orthogonal group is a differentiable morphism. If this composition were nontrivial it would be nontrivial on some simple factor of $G$. Therefore, there would be a noncompact simple Lie group contained in the orthogonal group. This is impossible because a semisimple Lie subgroup of a compact Lie group is closed, hence compact (cf. [2, p. 128]). Thus, the image of $f$ is contained in $\mathbb{R}^n$. This implies that $G$ has a nontrivial abelian factor, unless $f$ is the trivial morphism.

**Theorem.** A Riemannian globally symmetric space of noncompact type has no equivariant isometric immersions into $\mathbb{R}^n$.
Proof. Let \((f, \varphi)\) be an equivariant isometric immersion of \(M\). This means
(a) \(f\) is a differentiable group morphism from \(G\) into \(In\).
(b) \(\varphi\) is an isometric immersion from \(M\) into \(\mathbb{R}^n\) and
(c) \(\varphi(g(x)) = f(g)(\varphi(x))\) for every \(x \in M, g \in G\).

The Lemma implies that \(f(g)\) is the trivial linear transformation in \(\mathbb{R}^n\). Since \(G\) acts transitively, \(\varphi\) cannot be injective.

Note. (1) By means of the so-called class one representations of \(G\) it is easy to construct equivariant isometric immersions into infinite-dimensional Hilbert spaces.

(2) If \(M\) is a compact Riemannian homogeneous space, then there exist equivariant isometric immersions into \(\mathbb{R}^n\); cf. [3].

(3) If \(M\) is the upper half-plane with the Poincaré metric, then the result is due to [1].

References


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