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PSEUDOCOMPACT METACOMPACT SPACES ARE COMPACT

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In 1950, Arens and Dugundji [1] defined metacompact spaces and showed (A) countably compact metacompact spaces are compact. It was known by then that paracompact spaces are normal and that normal pseudocompact spaces are countably compact. Since paracompact spaces are metacompact, (A) also showed (B) pseudocompact paracompact spaces are compact. The results (A) and (B) raise the following question of Aull and others [2] as a common generalization: are pseudocompact metacompact spaces compact? This question was answered in the affirmative, first by Scott [3] and independently by Förster [4] and the author. The short proof given here also establishes an interesting property of Baire spaces.

All spaces are assumed completely regular. A space is *pseudocompact* if every continuous real-valued function on it is bounded. A space is *Baire* if no open set is the union of countably many nowhere dense sets. A *π-base* for a space X is a family $\mathfrak{B}$ of nonempty open subsets of X such that if G is a nonempty open subset of X, then some element of $\mathfrak{B}$ is contained in G.

LEMMA. *For any point-finite open cover $\mathfrak{U}$ of a Baire space X, there is a π-base $\mathfrak{B}$ for X, such that when $B \in \mathfrak{B}$ and $U \in \mathfrak{U}$ either B is contained in U or B is disjoint from U.*

PROOF. For each $n \in \mathbf{N}$, let $X_n = \{x \in X\colon x \text{ is in at most } n \text{ elements of } \mathfrak{U}\}$. Each X_n is closed and X is the union of $\{X_n\colon n \in \mathbf{N}\}$. Given an open subset G of X, G is a Baire subspace of X and thus, for some n, $G \cap X_n$ has nonempty interior. Let n_0 be the least such n. Let G' contained in G be an open subset of X which is contained in X_{n_0}. Let $x \in G'$ be in precisely n_0 elements of $\mathfrak{U}$ and let the intersection of these n_0 elements and G' be G''. G'' is an open subset of X contained in G and such that when $U \in \mathfrak{U}$ either G'' is contained in U or G'' is disjoint from U.

THEOREM. *Any pseudocompact metacompact space is compact.*

PROOF. Let $\mathfrak{O}$ be an open cover of X. Let $\mathfrak{U}$ be a point-finite refinement of $\mathfrak{O}$. Using regularity, assume without loss of generality that if $U \in \mathfrak{U}$ then the closure of U is contained in some element of $\mathfrak{O}$. Pseudocompact spaces are Baire (see, e.g., [5, p. 271]) and therefore the lemma applies here to yield a π-base $\mathfrak{B}$ for X, such that when $B \in \mathfrak{B}$ and $U \in \mathfrak{U}$ either B is contained in U or B is disjoint from U.

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Inductively define for each $n \in \mathbf{N}$, if possible, $A_n \in \mathfrak{B}$ such that each A_n is disjoint from the closure of the union of $\{U \in \mathfrak{U}: U \cap A_m \text{ is nonempty for some } m < n\}$. If the induction proceeds infinitely, then, since X is pseudocompact, there are no infinite discrete families of open sets (see, e.g., [5, p. 263]) and $\{A_n: n \in \mathbf{N}\}$ would have a limit point a, say $a \in U(a) \in \mathfrak{U}$, and $U(a)$ would intersect more than one A_n which contradicts their definition. Thus the induction stops at some n and since $\mathfrak{U}$ is point-finite, for each $m < n$, A_m is contained in finitely many elements of $\mathfrak{U}$ and thus A_m intersects finitely many elements of $\mathfrak{U}$. So there are finitely many elements of $\mathfrak{U}$ whose union is dense in X and therefore there are finitely many elements of $\mathfrak{O}$ which cover X as required.

References

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