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CARTAN MATRICES, FINITE GROUPS OF QUATERNIONS, AND KLEINIAN SINGULARITIES[1]

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To H.S.M. Coxeter for his 70th birthday

ABSTRACT. The eigenvectors of the Cartan matrices of affine type $\bar{A}_r$, $\bar{D}_r$, $\bar{E}_6$, $\bar{E}_7$, and $\bar{E}_8$ can be taken to be the columns of the character tables of the finite groups of quaternions.

PROPOSITION 1. *The Cartan matrices of affine type $\bar{A}_r$, $\bar{D}_r$, $\bar{E}_6$, $\bar{E}_7$, and $\bar{E}_8$ are positive semidefinite. Their eigenvectors can be taken to be the columns of the character table of the corresponding finite group of quaternions* [**3**, *Chapters* 6, 7], *namely the cyclic group Z_{r+1}, the binary dihedral* (*or dicyclic*) *group of order* $4r - 8$, *the binary tetrahedral, binary octahedral, and binary icosahedral group respectively.*

We remark that any matrix with such eigenvectors necessarily commutes with the matrices of the regular representation of the representation algebra of the appropriate finite group G. Only for $\bar{A}_0$, $\bar{A}_1$ and $\bar{E}_8$ are the eigenvalues simple and each eigenvector determined to within a scalar multiple.

For any (complex) representation R of G, we construct a graph Γ_R with the irreducible representations of G as its nodes and m_{jk} (possibly zero) directed edges from R_j to R_k where $R \otimes R_j = \bigoplus_k m_{jk} R_k$. We convene that an undirected edge between R_j and R_k represent the pair of directed edges from R_j to R_k and from R_k to R_j.

PROPOSITION 2. *Each of the five types of finite group described above has a faithful two-dimensional representation R_Q such that Γ_{R_Q} is the Coxeter graph of the corresponding affine type.*

Taking $R = R_Q$ we deduce that the Cartan matrix, C, of Γ_{R_Q} satisfies $C = 2I - M$ where $M = M_{R_Q} = (m_{jk})$. Proposition 1 follows as does the fact that the eigenvalues of M are the character values afforded by R_Q.

Singularities on algebraic varieties were studied by Schläfli in 1863 and by Cayley in 1869 although it seems that Du Val [**4**] was the first to relate the singularities to finite groups. His book [**5**, §5.40] contains a description of this relationship in terms of the topology of the spherical neighbourhood of the

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singularity. Steinberg [8, p. 156] writes: "Each singularity is realized naturally in the corresponding algebraic group, via a 'ridge' of singularities on the unipotent variety along its subregular subvariety." Brieskorn [1] remarks on the connection with finite groups which is discussed from an algebraic point of view in the forthcoming book of Slodowy [7].

ADDED IN PROOF. The connected undirected graphs with adjacency matrix having maximum eigenvalue 2 are precisely the Coxeter graphs above, together with the graph A_∞ which is the representation graph for SU_2 with $R = R_Q$, its natural two-dimensional representation. From this graph one may obtain each of the finite graphs as embeddings by restricting R to a finite subgroup of SU_2. An interpretation of the dual of the A_∞ graph as the Dynkin curve of an appropriate singularity is wanting.

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