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A VARIANT OF THE CHAIN RULE FOR DIFFERENTIAL CALCULUS

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ABSTRACT. A version of the chain rule is developed which can be applied to the construction of solutions to quasi-linear hyperbolic partial differential equations.

In this note we present a variant of the usual chain rule for differential calculus which is extremely useful in the demonstration (see [3]) that certain types of quasi-linear unbounded vector fields generate continuous flows. We consider a composition $f \circ \alpha$, where f maps an open subset of a Banach space Y to a Banach space Z, and α is a curve whose image is contained in the domain of f. By strengthening the assumptions on the differentiability of f, we can weaken the assumptions on α to something less than continuity in Y.

We make the following assumptions throughout this article: X, Y, and Z are Banach spaces, with Y continuously and densely included in X; V is an open subset of Y, and f: $V \to Z$; $[a, b] \subset \mathbf{R}$, and α: $[a, b] \to V$. $B(X, Z)$ will denote the space of continuous linear maps from X to Z, $B(Y, Z)$ the space of continuous linear maps from Y to Z. If $l \in B(Y, Z)$ has an extension to an element of $B(X, Z)$, then we will use the same symbol ("l", in this case) to denote the extension, and $\|l\|_{X,Z}$ will denote the norm of the extension.

LEMMA. *Let $t \in [a, b]$, and assume that α: $[a, b] \to X$ is differentiable at t. Assume that f has a Gateaux derivative at $\alpha(t)$, and that $Df(\alpha(t))$ extends to an element of $B(X, Z)$. Assume in addition that there exists $k > 0$ such that $\|f(v_2) - f(v_1)\|_Z \leq k\|v_2 - v_1\|_X$ for each $v_1, v_2 \in V$. Then $f \circ \alpha$ is differentiable at t, and $(f \circ \alpha)'(t) = Df(\alpha(t))(\alpha'(t))$.*

PROOF. Since Y is dense in X, there exists a sequence $\{y_n\}_{n \in N}$ of elements of Y which converges in X to $\alpha'(t)$. For each $n \in N$ and $h \neq 0$,

$$\begin{aligned}
&\|h^{-1}[f(\alpha(t+h)) - f(\alpha(t))] - Df(\alpha(t))(\alpha'(t))\|_Z \\
&\quad \leq \|h^{-1}[f(\alpha(t+h)) - f(\alpha(t) + hy_n)]\|_Z \\
&\qquad + \|h^{-1}[f(\alpha(t) + hy_n) - f(\alpha(t))] - Df(\alpha(t))(y_n)\|_Z \\
&\qquad + \|Df(\alpha(t))(y_n - \alpha'(t))\|_Z.
\end{aligned}$$

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Now,

$$\begin{aligned}\|h^{-1}[f(\alpha(t+h)) - f(\alpha(t)+hy_n)]\|_Z &= |h|^{-1}\|f(\alpha(t+h)) - f(\alpha(t)+hy_n)\|_Z \\ &\leq k|h|^{-1}\|\alpha(t+h) - \alpha(t) - hy_n\|_X \\ &\leq k|h|^{-1}\|\alpha(t+h) - \alpha(t) - h\alpha'(t)\|_X + k\|\alpha'(t) - y_n\|_X,\end{aligned}$$

and

$$\|Df(\alpha(t))(y_n - \alpha'(t))\|_Z \leq \|Df(\alpha(t))\|_{X,Z}\|y_n - \alpha'(t)\|_X.$$

Thus, for each $n \in N$,

$$\begin{aligned}\limsup_{|h|\to 0} \|h^{-1}[f(\alpha(t+h)) - f(\alpha(t))] - Df(\alpha(t))(\alpha'(t))\|_Z \\ \leq (k + \|Df(\alpha(t))\|_{X,Z})\|y_n - \alpha'(t)\|_X.\end{aligned}$$

Since $\|y_n - \alpha'(t)\|_X \to 0$ as $n \to \infty$, the lemma is proved. □

THEOREM. *Assume that V is convex, that f has a Gateaux derivative at each point of V, that each $Df(v)$ has an extension to an element of $B(X, Z)$, and that $Df(V)$ is a bounded subset of $B(X, Z)$. Assume in addition that $\alpha(\cdot)$ is absolutely continuous and differentiable almost everywhere from $[a, b]$ to X. Then $f \circ \alpha$ is absolutely continuous and differentiable almost everywhere from $[a, b]$ to Z, and $(f \circ \alpha)'(t) = Df(\alpha(t))(\alpha'(t))$ for each t at which $\alpha'(t)$ exists.*

PROOF. Choose $c > 0$ such that $\|Df(v)\|_{X,Z} \leq c$ for every $v \in V$. From the Mean Value Theorem it follows that $\|f(v_2) - f(v_1)\|_Z \leq c\|v_2 - v_1\|_X$ for each v_1, $v_2 \in V$. The above lemma then implies that $(f \circ \alpha)'(t) = Df(\alpha(t))(\alpha'(t))$ for each t at which $\alpha'(t)$ exists. Thus, $f \circ \alpha$ is differentiable almost everywhere. Since $\alpha(\cdot)$ is absolutely continuous from $[a, b]$ to X, the estimate $\|f(v_2) - f(v_1)\|_Z \leq c\|v_2 - v_1\|_X$ for every $v_1, v_2 \in V$ implies that $f \circ \alpha$ is absolutely continuous from $[a, b]$ to Z. □

It is easy to use our lemma to produce versions of the above theorem in which the domain of α is permitted to be an open subset of an arbitrary Banach space and α is assumed to be Gateaux differentiable when regarded as a map into X (cf. the chain rule for β-differentiability in [1, §5]). However, many such generalizations are possible and so, having no specific application in mind for such a generalization, we have chosen to restrict our attention in this note to a version of demonstrated usefulness.

The proof of our lemma is an adaptation of the proof of a weaker result which appears in [2].

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