GROUP RINGS WHOSE TORSION UNITS FORM A SUBGROUP

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Abstract. Denote by $TU(ZG)$ the set of units of finite order of the integral group ring of a group $G$. We determine the class of all groups $G$ such that $TU(ZG)$ is a subgroup and study how this property relates to certain properties of the unit groups.

1. Introduction. Let $G$ be a group. We denote by $ZG$ its integral group ring and by $U(ZG)$ the group of units of this ring. Also, we shall denote by $T = T(G)$ and $TU(ZG)$ the set of all elements of finite order in $G$ and $U(ZG)$ respectively.

S. K. Sehgal and H. J. Zassenhaus have determined the classes of all groups $G$ such that $U(ZG)$ is a nilpotent or an FC group in [3] and [4]. If restricted to finite groups, it is easy to see that both classes coincide.

M. M. Parmenter and C. Polcino Milies have shown in [1] that, in the finite case, the characterization of such groups $G$ follows from the fact that, in both cases, $TU(ZG)$ forms a subgroup.

In this note, we determine all groups $G$ which are either nilpotent or FC and such that $TU(ZG)$ is a subgroup. It will follow that this property is not enough to lead either to nilpotency or the FC property when $G$ is infinite.

2. The Theorem. We begin with

Lemma. Let $G$ be a group such that $TU(ZG)$ forms a subgroup. Then $TU(ZG) = \pm T$, i.e. every unit of finite order is trivial.

Proof. Let $u \in TU(ZG)$ with $o(u) = p_1^{n_1} \ldots p_k^{n_k}$. We shall show that $u \in \pm T$ by induction on $t$.

If $t = 1$ then $u$ is a $p$-element for some rational prime $p$; hence, there is an element $g \in supp(u)$ such that $o(g)$ is finite [2, Theorem VI.2.1]. Since $TU(ZG)$ is a subgroup, $v = g^{-1} \cdot u$ is a unit of finite order and, writing $v = \sum_{a \in G} v(a)a$, we have that $v(1) \neq 0$. It follows from [2, Corollary II.1.2] that $v = \pm 1$, thus $u = \pm g \in T$.

Now, we assume that the result holds if $o(u)$ is divisible at most by $t - 1$ different primes and let $o(u) = p_1^{n_1} \ldots p_t^{n_t}$.
Writing \( o(u) = mp_i^n \) where \( \gcd(m, p_i^n) = 1 \), there exist integers \( r, s \) such that \( m + sp_i^n = 1 \). Set \( u_1 = u^{rm} \) and \( u_2 = u^{sp_i^n} \). Then \( u = u_1 \cdot u_2 \) and \( u_1, u_2 \in \pm T \) by the induction hypothesis. □

We can now prove our main result.

**Theorem.** Let \( G \) be a group such that \( TU(ZG) \) is a subgroup of \( U(ZG) \). Then \( T \) is a subgroup of \( G \) and one of the following conditions holds.

(i) \( T \) is abelian and, for all \( x \in G \) and all \( t \in T \), we have that \( x^{-1}tx = t^r \), where \( r = i(x) \) locally.

(ii) \( T = K_8 \times E \) where \( K_8 \) is the quaternion group of order 8 and \( E \) is an elementary abelian 2-group. Furthermore, \( E \) is central in \( G \) and conjugation by a fixed element of \( G \) induces in \( K_8 \) one of its four inner automorphisms.

Conversely, if \( G \) is a nilpotent or FC group satisfying (i) or (ii), then \( TU(ZG) \) is a subgroup of \( U(ZG) \).

**Proof.** First assume that \( TU(ZG) \) is a subgroup. It follows from the Lemma that \( TU(ZG) = \pm T \), hence [1] or [2, Theorem II.4.1] shows that \( T \) is either abelian or \( T = K_8 \times E \) as in (ii).

For elements \( t \in T \), and \( x \in G \) we consider \( u = 1 + (t - 1)xt \) where \( t = 1 + t_1 + \cdots + t_\infty \).

Then \( u^{-1} = 1 - (t - 1)xt \) and

\[
uttu^{-1} = t + xt^\ast -2txt^\ast + t^2xt^\ast.
\]

Since \( utu^{-1} \in TU(ZT) \) we must have \( txt = xt^\ast \) or \( txt = t^2xt^\ast \) and consequently \( t^\ast \in \langle t \rangle \).

If \( T \) is abelian, the last equality shows that (i) holds. If \( T = K_8 \times E \), it readily implies that \( E \) is central and that the automorphism induced in \( K_8 \) is inner.

Finally assume that either (i) or (ii) holds. From [2, Corollary VI.1.24] we know that \( QG \) contains no nonzero nilpotent elements; hence, every idempotent is central and [2, Lemma VI.3.22] shows that \( U(ZG) = U(ZT) \cdot G \).

Now, we claim that if \( u \in U(ZG) \) is an element of finite order, then \( u \in U(ZT) \).

Assume that \( u = vg \) where \( v \in U(ZT) \) and \( g \in G \) is an element of infinite order. Since \( gv = v'g \) for some \( v' \in U(ZT) \), if \( n = o(u) \) we have that \( 1 = u^n = v^n g^n \), with \( v^n \in U(ZT) \). Consequently \( g^n \in U(ZT) \), a contradiction.

We have thus shown that \( TU(ZG) \subset (ZT) \).

If (i) holds, \( U(ZT) \) is abelian and the result follows trivially.

If (ii) holds, from [2, Corollary II.2.5] we have that \( U(ZT) = \pm T \), hence \( U(ZG) = \pm G \) and thus \( TU(ZG) = \pm T \) is a subgroup. □

As a consequence of the Theorem and a result of Sehgal [2, Theorem VI.1.20] we have

**Corollary.** Let \( G \) be a nilpotent or FC group such that \( TU(ZG) \) forms a subgroup. Then \( ZG \) contains no nilpotent elements.

3. **An example.** We conclude with an example showing that if \( G \) is infinite nilpotent or FC then the fact that \( TU(ZG) \) forms a subgroup does not imply that \( U(ZG) \) is either nilpotent or FC.
Consider the group $G = \langle t, x | t^9 = 1, xtx^{-1} = t^7 \rangle$. Denote $(g_1, g_2) = g_1g_2g_1^{-1}g_2^{-1}$, $G^{(1)} = \langle (g_1, g_2) | g_1, g_2 \in G \rangle$ and $G^{(n)} = \langle (x, g) | x \in G^{(n-1)}, g \in G \rangle$.

It is easy to see that $G^{(1)} \subset \langle t \rangle$.

Now, if $g = x^k t^m$ we have

$$(g, t^n) = x^k t^n x^{-k} t^{-m} = t^{n(7^k - 1)} = t^{6n(7^{k-1} + 7^{k-2} + \cdots + 1)} \in \langle t^3 \rangle.$$ 

In a similar way it follows that $G^{(3)} = \langle 1 \rangle$. Thus $G$ is nilpotent.

Also, if $C(g)$ denotes the conjugacy class of an element $g \in G$, it is easy to see that $C(g) \subset G^{(1)} g = \langle t \rangle g$ and is thus finite.

Consequently, $G$ is also an FC group.

Hence, $G$ satisfies the conditions in our theorem and is both nilpotent and FC, but it follows from Theorems VI.3.23 and VI.5.3 in [2] that $U(ZG)$ is neither nilpotent nor FC.

REFERENCES