PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 2, February 1981

INTERSECTIONS OF PURE SUBGROUPS IN ABELIAN GROUPS

D. BOYER AND K. M. RANGASWAMY

ABSTRACT. It is shown that a subgroup H of an abelian group G is an intersection of pure subgroups of G if and only if, for all primes p and positive integers n, $p^n g \in H$ and $p^{n-1}g \notin H$ imply that there exists $z \in G$ such that $p^n z = 0$ and $p^{n-1}z \notin H$. This solves a problem posed by L. Fuchs in [2] and [3].

Introduction. B. Charles [1] and S. Khabbaz [6] independently obtained characterizations of intersections of divisible subgroups of a divisible abelian group. Since divisible subgroups are always pure, L. Fuchs, [2], [3], asked for a characterization of those subgroups of an abelian group that are an intersection of pure subgroups. C. Megibben [7] answered this when the group is a p-group. In the present paper we answer the question for arbitrary abelian groups. Our results also throw some light on Problem 13 of Fuchs [4].

We follow the idea of Megibben in realizing the given subgroup as an intersection of neat subgroups which are then shown to be pure. All groups in this paper are abelian, and we shall follow essentially the notation and terminology of [4]. All topological concepts refer to the p-adic topology. A subgroup H of a group G is *p-neat* if $H \cap pG = pH$. A minimal neat subgroup N containing a subgroup S is called a *neat hull* of S.

We list the following two conditions on a subgroup H of a group G for ready reference.

(∗) For all primes p and all positive integers n, $p^n g \in H$ and $p^{n-1}g \notin H$ imply there exists $z \in G$ such that $p^n z = 0$ and $p^{n-1}z \notin H$.

(M) For all primes p and all positive integers n, $p^n G[p] \subset H$ implies $p^n G_p \subset H$.

REMARK. Megibben [7] proved that, for p-groups G, the condition (M) characterizes intersections of pure subgroups of G. However, in general, a subgroup with condition (M) need not be an intersection of pure subgroups since, for example, in a torsion free group every subgroup satisfies (M). Our main theorem states that subgroups with condition (∗) are the intersections of pure subgroups. We wish to note that condition (∗) implies condition (M), for suppose $p^n G_p \not\subset H$ and $p^n g$ is an element of the least order such that $p^n g \notin H$, then $p^{n+1}g \in H$ and therefore, by (∗), G contains an element z with $p^{n+1}z = 0$ and $p^n z \notin H$; thus $p^n G[p] \not\subset H$. As in the example above, condition (M) does not imply condition (∗) since subgroups of a torsion free group satisfying (∗) are easily seen to be pure.

Received by the editors April 21, 1980; presented to the Society, January 3, 1980.
1980 *Mathematics Subject Classification*. Primary 20K99.
Key words and phrases. Abelian group, pure subgroup.

Finally, in a p-group G, the two conditions are equivalent. Indeed, if there is no z with $p^n z = 0$ and $p^{n-1}z \notin H$, then $p^{n-1}G[p] \subset H$; hence, by (M), $p^{n-1}G \subset H$, and there is no $g \in G$ such that $p^n g \in H$ and $p^{n-1}g \notin H$.

We will need the following two lemmas:

LEMMA 1 [5]. *If the subgroup H of the group G is p-neat in G, and $H[p]$ is dense in $G[p]$, then H is p-pure in G.*

LEMMA 2 [8]. *A subgroup S of the group G is an intersection of neat subgroups (indeed, of all the neat hulls of S) if and only if, for each prime p, $S[p] \neq G[p]$ whenever $(G/S)[p] \neq 0$.*

THEOREM. *A subgroup H of the group G is an intersection of pure subgroups if and only if H satisfies condition* $(*)$.

PROOF. Since the necessity of $(*)$ is clear, we prove that $(*)$ is sufficient. Suppose H has property $(*)$. Let P be the set of primes relevant to G, and let P' be the set of those primes $p \in P$ such that $p^{n_p}G_p \subset H$, for some n_p. Choosing n_p minimally, we can assume that $p^{n_p-1}G_p \not\subset H$, whenever $n_p > 0$. By the remark, H satisfies condition (M); hence, for the primes $p \in P'$ for which $n_p > 0, p^{n_p-1}G[p] \not\subset H$. Thus for all $p \in P'$ there is an element a_p of order p^{n_p} such that $H \cap \langle a_p \rangle = 0$. Let $T = \bigoplus_{p\in P'} p^{n_p}G_p$. Then H/T satisfies $(*)$ in G/T and we show first that H/T is an intersection of pure subgroups of G/T. For convenience of expression we write G and H in place of G/T and H/T, respectively, with the additional assumption that, for all $p \in P'$, $p^{n_p}G_p = 0$ and $H \cap \langle a_p \rangle = 0$, where $\langle a_p \rangle$ is (an absolute summand of G_p) of order p^{n_p}.

For each $p \in P$, we choose subgroups $A_p \oplus B_p$ as follows: (i) If the reduced part of G_p is bounded, then let $A_p[p] = H[p]$ and $G_p = D_p \oplus A_p \oplus B_p$, where $D_p = \langle a_p \rangle$ if $p \in P'$, and if $p \notin P'$, then D_p is a subgroup maximal with respect to being divisible and having $D_p \cap H = 0$. Such a D_p exists since by $(*)$ and (M), whenever $p \notin P'$, $p^nG[p] \not\subset H$, for all n. (ii) If the reduced part of G_p is unbounded, then choose $A_p \oplus B_p = S$, a basic subgroup of G_p with $A_p[p] = S[p] \cap H$, so that $H[p] \oplus B_p[p]$ is dense in $G[p]$. Here we can assume that $H[p] \oplus B_p[p] \neq G[p]$, for if $H[p] \oplus B_p[p] = G[p]$, and $H[p] \neq \overline{H[p]}$, its closure in $G[p]$, then we can replace B_p by a summand B_1, where $B_p[p] = (\overline{H[p]} \cap B_p[p]) \oplus B_1[p]$; and if $H[p] = \overline{H[p]}$, then since $p^nG[p] \not\subset H$ for all n, B_p must be unbounded and we can replace B_p by a proper basic subgroup B_1 of B_p, so that in either case, $H[p] \oplus B_1[p]$ is dense and properly contained in $G[p]$.

Since the subgroup $H \oplus \bigoplus_{p\in P} B_p[p]$ satisfies the hypothesis of Lemma 2, it is an intersection of neat subgroups S, which can be chosen to be neat hulls of $H \oplus \bigoplus_{p\in P} B_p[p]$. We show that each such S is pure. By Lemma 1, S is p-pure for all $p \notin P'$. Suppose $p \in P'$ with $n_p > 0$, so that $G[p] = p^{n_p-1}\langle a_p \rangle \oplus S[p]$. Then by the p-neatness of S and a simple induction, $S \cap (p^iG) = p^iS$ for all $i < n_p$, whence, S_p is pure in G_p and, indeed, $\langle a_p \rangle \oplus S_p = G_p$. Suppose $p^kg \in S$ for some $k > n_p$. Since $H \oplus \bigoplus_{p\in P} B[p]$ is an essential subgroup of its neat hull S, we can assume without loss of generality that $p^rg \in H$ for some $r > k$. Since H satisfies $(*)$

and $p^{n_p}G_p = 0$, it readily follows that $p^{n_p}g \in H$. Then $p^{n_p}g \in S \cap p^{n_p}G = p^{n_p}S$, whence $p^kg = p^{k-n_p}(p^{n_p}g) \in p^kS$. This proves the p-purity of S for all $p \in P'$. Thus S is pure in G.

We now claim that H is the intersection of the subgroups $H \oplus \bigoplus_{p\in P} B_p[p]$, where, for each $p \in P$, $A_p \oplus B_p$ has been chosen with the prescribed properties. Suppose $x \in H \oplus \bigoplus_{p\in P} B_p[p]$ and $x \notin H$, say $x = h + b_1 + \cdots + b_n$, $h \in H$, $b_i \in B_{p_i}$, $i = 1, \ldots, n$, with $b_i \neq 0$ for some i. Write $B_{p_i}[p_i] = \langle b_i\rangle \oplus C_i$. Choose $z_i \in G[p_i]$ with $z_i \notin H + B_{p_i}[p_i]$ and $h_{p_i}(z_i) \geq h(b_i)$, where the equality holds only if $p_i \in P'$ and $h(b_i) = n_{p_i} - 1$. Construct B'_{p_i} such that $B'_{p_i}[p_i] = \langle z_i + b_i\rangle \oplus C_i$ and $A_{p_i} \oplus B'_{p_i}$ satisfies the properties mentioned in cases (i) and (ii). Now clearly

$$x \notin H + B'_{p_i}[p_i] \oplus \bigoplus_{\substack{q\in P\\ q\neq p_i}} B[q].$$

Thus we conclude that H is the intersection of the pure subgroups S.

Returning to our original notation, we have now that H/T is an intersection of pure subgroups S/T, so that H is the intersection of the subgroups S. Thus we are done if we show that the subgroups S are pure in G. For all $p \in P'$, $G_p/T_p = (\langle a_p\rangle + T_p)/T_p \oplus S_p/T_p$, hence $G_p = \langle a_p\rangle \oplus S_p$. Thus $\bigoplus_{p\in P'} S_p$ is pure in G, $T \subset \bigoplus_{p\in P'} S_p$, and S/T is pure in G; hence S is pure in G.

REMARK. It is well-known that a subgroup H of a group G is pure if and only if every coset of G mod H contains an element of the same order as that coset. The condition (*) can be formulated in terms of cosets as follows: For all primes p and all positive integers n, if there is a coset $g + H$ of order p^n, then there is another coset $z + H$ of order p^n containing an element of order p^n.

REFERENCES

1. B. Charles, *Une caractérisation des intersections de sous-groups divisibles*, C. R. Acad. Sci. Paris Ser. A-B **250** (1960), 256–257.

2. L. Fuchs, Review of [1], #64, Math. Reviews **22** (1961), 10–11.

3. ______, *Recent results and problems on abelian groups*, Topics in Abelian Groups, Scott, Foresman and Co., Glenview, Ill., 1963, pp. 9–40.

4. ______, *Infinite abelian groups*. I, Academic Press, New York, 1970.

5. P. Hill and C. Megibben, *Minimal pure subgroups in primary groups*, Bull. Soc. Math. France **92** (1964), 251–257.

6. S. Khabbaz, *The subgroups of a divisible group G which can be represented as intersections of divisible subgroups of G*, Pacific J. Math. **11** (1961), 267–273.

7. C. Megibben, *On subgroups of primary abelian groups*, Publ. Math. Debrecen **12** (1965), 293–294.

8. K. M. Rangaswamy, *Characterization of intersections of neat subgroups of abelian groups*, J. Indian Math. Soc. **29** (1965), 31–36.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS AT EL PASO, EL PASO, TEXAS 79968