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THE RESTRICTION OF WHITTAKER MODULES TO CERTAIN PARABOLIC SUBALGEBRAS

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ABSTRACT. Certain twisted cohomology spaces of modules for the standard maximal parabolic subalgebra of $\mathfrak{sl}(n, \mathbf{C})$ are studied. These results are shown to imply new proofs of results of Kostant on Whittaker modules in the special case of real forms of products of $\mathfrak{sl}(n, \mathbf{C})$'s.

1. Introduction. In a major recent paper [3], Kostant described the structure of Whittaker modules and Whittaker vectors for quasisplit, linear semisimple Lie groups. In the case of $\mathrm{SL}(n, \mathbf{R})$ a substantial portion of this work had been done independently by Casselman and Zuckerman.

In Kostant's paper there were two major results from which everything else in the paper followed. The first was the irreducibility of certain universal modules. The second was the vanishing of certain cohomology spaces.

In this note we give new proofs of the theorems of Kostant (and therefore those of Casselman and Zuckerman) in the case when G is a real form of a product of groups isomorphic with $\mathrm{SL}(n, \mathbf{C})$. We prove a generalization of Kostant's vanishing theorem which is applicable to certain parabolic subalgebras of $\times_{i=1}^{k} \mathfrak{sl}(n_i, \mathbf{C})$. In particular, we show that Kostant's universal modules are already irreducible when restricted to these parabolic subalgebras.

Theorem 3.2 is quite general. It is an important special case of a theorem of Kostant. Its proof is included since it is new and involves the theory of Verma modules.

This paper is a spin-off from joint work, [2], of the author with Roe Goodman. In [2], one can find applications of the work of Kostant, hence of the material in this paper. Philosophically this paper is related to a joint paper of the author with J. T. Stafford [5] which studies the analogous questions for the case of the trivial character of $\mathfrak{n}$.

We note that our definition of Whittaker modules is weaker than Kostant's. We make *no assumption of finite generation in our results on vanishing of cohomology*.

2. The study of certain modules. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbf{C}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. Let Δ be the root system of $(\mathfrak{g}, \mathfrak{h})$. Fix Δ^+ to be a system of positive roots for Δ. Let $\mathfrak{n} = \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$, $\bar{\mathfrak{n}} = \sum_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}$. Set $\bar{\mathfrak{b}} = \mathfrak{h} \oplus \bar{\mathfrak{n}}$.

Let $\psi\colon \mathfrak{n} \to \mathbf{C}$ be a Lie algebra homomorphism. We assume that ψ is generic. That is if $\alpha \in \Delta^+$ is simple then $\psi|\mathfrak{g}_\alpha \neq 0$.

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We look upon $U(\mathfrak{h})$ as a module for $\bar{\mathfrak{b}}$ with action, $X \cdot U(\mathfrak{h}) = 0$, $X \in \bar{\mathfrak{n}}$ and $H \in \mathfrak{h}$ acts by multiplication. Set

$$V = \mathrm{Hom}_{U(\bar{\mathfrak{b}})}(U(\mathfrak{g}), U(\mathfrak{h})).$$

Here $U(\mathfrak{g})$ is a left $U(\bar{\mathfrak{b}})$-module under left multiplication and a right $U(\mathfrak{g})$-module under right multiplication.

Let ε: $U(\mathfrak{g}) \to \mathbf{C}$ be the augmentation (i.e. $\varepsilon(1) = 1$ and $\varepsilon(\mathfrak{g}U(\mathfrak{g})) = 0$). Let $\omega \in V$ be defined by $\omega(\bar{n}hn) = \varepsilon(\bar{n})\psi(n)h$ for $\bar{n} \in U(\bar{\mathfrak{n}})$, $h \in U(\mathfrak{h})$, $n \in U(\mathfrak{n})$.

Set $W_\psi = U(\mathfrak{g})\omega$. Set $V_{\psi,k} = \{f \in V | (X - \psi(X))^k f = 0, X \in \mathfrak{n}\}$. Then it is easily checked that $V_\psi = \bigcup_{k=1}^{\infty} V_{\psi,k}$ is a $\mathfrak{g}$-submodule of V and that $W_\psi \subset V_\psi$ is a $\mathfrak{g}$-submodule.

Let $\rho = (1/2)\Sigma_{\alpha \in \Delta^+} \alpha$. If $\Lambda \in h^*$ let $\mathbf{C}_\Lambda$ be the $\bar{\mathfrak{b}}$-module, $\mathbf{C}$, with $\bar{\mathfrak{n}} \cdot \mathbf{C} = 0$ and $H \cdot 1 = \Lambda(H) \cdot 1$, $H \in \mathfrak{h}$.

Define B_Λ: $\mathrm{Hom}_{U(\bar{\mathfrak{b}})}(U(\mathfrak{g}), U(\mathfrak{h})) \to \mathrm{Hom}_{U(\bar{\mathfrak{b}})}(U(\mathfrak{g}), \mathbf{C}_{\Lambda-\rho})$ by $B_\Lambda(f)(g) = (\Lambda - \rho)(f(g))$, $g \in U(\mathfrak{g})$. Then B_Λ is a surjective $\mathfrak{g}$-module homomorphism.

Let ξ: $U(\mathfrak{g}) \to U(\mathfrak{h})$ be defined by $\xi(nh\bar{n}) = \varepsilon(n)\varepsilon(\bar{n})h$. Then $\xi \in V$. We set $\xi_\Lambda = B_\Lambda(\xi)$. Define for $\bar{n} \in U(\bar{\mathfrak{n}})$, and $n \in U(\mathfrak{n})$, $\langle n, \bar{n}\rangle_\Lambda = \xi_\Lambda(n\bar{n})$. Then $\langle \mathrm{ad}\, H \cdot n, \bar{n}\rangle_\Lambda = -\langle n, \mathrm{ad}\, H \cdot \bar{n}\rangle_\Lambda$, $H \in \mathfrak{h}$. Let $\mathcal{C} = \{\Lambda \in \mathfrak{h}^* | \langle\ ,\ \rangle_\Lambda$ is nondegenerate$\}$. Then as is well known, $\mathcal{C}$ is Zariski-dense in $\mathfrak{h}^*$.

Let $\alpha_1, \ldots, \alpha_l$ be the simple roots in Δ^+. Let $H_0 \in \mathfrak{h}$ be defined by $\alpha_i(H_0) = 1$, $i = 1, \ldots, l$. Then $U(\mathfrak{n}) = \bigoplus_{j=0}^{\infty} U(\mathfrak{n})_j$ with $U(\mathfrak{n})_j = \{n \in U(\mathfrak{n}) | \mathrm{ad}\, H_0 \cdot n = j \cdot n\}$. Also, $U(\bar{\mathfrak{n}}) = \bigoplus_{j=0}^{\infty} U(\bar{\mathfrak{n}})_{-j}$ with $U(\bar{\mathfrak{n}})_{-j} = \{\bar{n} \in U(\bar{\mathfrak{n}}) | \mathrm{ad}\, H_0 \cdot \bar{n} = -j\bar{n}\}$.

As a $U(\mathfrak{n})$-module $\mathrm{Hom}_{U(\bar{\mathfrak{b}})}(U(\mathfrak{g}), \mathbf{C}_{\Lambda-\rho})$ is just $U(\mathfrak{n})^*$. Hence there is a unique element $\omega_\Lambda \in \mathrm{Hom}_{U(\bar{\mathfrak{b}})}(U(\mathfrak{g}), \mathbf{C}_{\Lambda-\rho})$ such that $X \cdot \omega_\Lambda = \psi(X)\omega_\Lambda$. Clearly, $\omega_\Lambda = B_\Lambda(\omega)$. For $\Lambda \in \mathcal{C}$, $j \in \mathbf{Z}$, $j > 0$, let $\bar{n}_j(\Lambda)$ be the unique element of $U(\bar{n})_{-j}$ such that $\langle n, \bar{n}_j(\Lambda)\rangle_\Lambda = \psi(n)$ for $n \in U(\mathfrak{n})_j$. Then, if $\Lambda \in \mathcal{C}$, $\omega_\Lambda(g) = \Sigma_j \xi_\Lambda(g\bar{n}_j(\Lambda))$. (For each fixed $g \in U(\mathfrak{g})$ this sum is finite.)

Let $M_\psi = U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} \mathbf{C}^\psi$ where $\mathbf{C}^\psi$ is the $\mathfrak{n}$-module $\mathbf{C}$ with $X \cdot 1 = \psi(X) \cdot 1$.

LEMMA 2.1. *The natural $\mathfrak{g}$-homomorphism $M_\psi \to W_\psi$ given by $1 \otimes 1 \to \omega$ is bijective.*

PROOF. As a $\bar{\mathfrak{b}}$-module, $M_\psi = U(\bar{\mathfrak{b}})$ under left translation. We must therefore show that if $\bar{b} \in U(\bar{\mathfrak{b}})$ then $\bar{b} \cdot \omega = 0$ implies $\bar{b} = 0$.

Let $\bar{b} = \Sigma_{i=1}^d \bar{n}^i h_i$ with $\bar{n}^i \in U(\bar{\mathfrak{n}})$ and $h_i \in U(\mathfrak{h})$, $h_1, \ldots, h_d$ linearly independent. We show that if $\bar{b} \cdot \omega = 0$ then $\bar{n}^i = 0$, $i = 1, \ldots, d$. We assume $\bar{b} \neq 0$ is as above and $\bar{b} \cdot \omega = 0$.

If $\bar{n} \in U(\bar{\mathfrak{n}})$ then $\bar{n} = \Sigma_{j=0}^k \bar{n}_j$ with $\bar{n}_j \in U(\bar{\mathfrak{n}})_{-j}$. Let k_0 be minimal among the $k > 0$ such that $(\bar{n}^i)_k \neq 0$ for some $1 < i < d$. Let $\Lambda \in \mathcal{C}$.

If $n \in U(\mathfrak{n})$ then

$$0 = B_\Lambda(\bar{b} \cdot \omega)(n) = \sum_j \xi_\Lambda(n\,\overline{bn}_j(\Lambda))$$

$$= \sum_{j,i} \xi_\Lambda(n\bar{n}^i h_i \bar{n}_j(\Lambda)).$$

For $\mu \in \mathfrak{h}^*$, let $U(\bar{\mathfrak{n}})_{-\mu} = \{\bar{n} \in U(\bar{\mathfrak{n}}) | \operatorname{ad} H \cdot \bar{n} = -\mu(H)\bar{n}, H \in \mathfrak{h}\}$. Then $\bar{n}_j(\Lambda) = \sum_\mu \bar{n}_j(\Lambda)_\mu$, $\bar{n}_j(\Lambda)_\mu \in U(\bar{\mathfrak{n}})_{-\mu}$. Thus

$$0 = \sum_{j,i,\mu} \xi_\Lambda\big(n\bar{n}^i\bar{n}_j(\Lambda)_\mu\big)(\Lambda - \rho - \mu)(h_i).$$

Fix $n \in U(\mathfrak{n})_{k_0}$. Then

$$\begin{aligned} 0 &= \sum_{j,i,\mu} \xi_\Lambda\big(n(n^i)_{k_0}n_j(\Lambda)_\mu\big)(\Lambda - \rho - \mu)(h_i) \\ &= \sum_i \xi_\Lambda\big(n(\bar{n}^i)_{k_0}\big)(\Lambda - \rho)(h_i). \end{aligned}$$

The last equality is seen by comparing weights and noting that $\bar{n}_0(\Lambda) = 1$. We therefore have (1) If $\Lambda \in \mathcal{C}$ then

$$\sum_{i=1}^{d} (\Lambda - \rho)(h_i)(\bar{n}^i)_{k_0} = 0.$$

(1) combined with the fact that $\mathcal{C}$ is Zariski-dense in $\mathfrak{h}^*$ implies that $(\bar{n}^i)_{k_0} = 0$ for all i. This contradicts the definition of k_0, and proves the result.

For $h_0 \in U(\mathfrak{h})$, let ${}_{h_0}\omega(\bar{n}hn) = \varepsilon(\bar{n})\psi(\bar{n})h_0h$. Then ${}_{h_0}\omega \in V_{\psi,0}$ for $h_0 \in U(\mathfrak{h})$.

LEMMA 2.2. (1) *If $f \in V_{\psi,0}$ then $f =_{f(1)} \omega$.*

(2) *If $Z \in \mathfrak{z}(\mathfrak{g})$ (the center of $U(\mathfrak{g})$) then $Z \cdot \omega =_{\xi(Z)} \omega$.*

PROOF. If $f \in V_{\psi,0}$ then $f(\bar{n}hn) = \psi(n)\varepsilon(\bar{n})hf(1) =_{f(1)} \omega(\bar{n}hn)$. This proves (1).

To prove (2) we note that

$$Z = \xi(Z) + \sum_{p>0,q,r,s} \bar{n}_{q,p}h_rn_{r,p}$$

with $\bar{n}_{q,p} \in U(\bar{\mathfrak{n}})_{-p}$, $n_{r,p} \in U(\mathfrak{n})_p$. Now $f = Z \cdot \omega \subset V_{\psi,0}$. Hence $f =_{f(1)} \omega$.

$$\begin{aligned} f(1) &= \xi(Z) + \sum_{p>0,q,r,s} \omega\big(\bar{n}_{q,p}h_rn_{r,p}\big) \\ &= \xi(Z) + \sum_{p>0,q,r,s} \varepsilon\big(\bar{n}_{q,p}\big)h_r\omega(n_{r,p}) \\ &= \xi(Z) \quad \text{since } \varepsilon\big(\bar{n}_{q,p}\big) = 0, p > 0. \end{aligned}$$

This proves (2).

If M is a $\mathfrak{g}$-module set $\operatorname{Wh}(M) = \{m \in M | X \cdot m = \psi(X)m, X \in \mathfrak{n}\}$.

THEOREM 2.3. (Compare Kostant [3, Theorem 3.3].) $\operatorname{Wh}(M_\psi) = \mathfrak{z}(\mathfrak{g})(1 \otimes 1)$.

PROOF. If $\Lambda \in \mathfrak{h}^*$ define $\overline{M}^\Lambda = U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{b}})} \mathbf{C}_{\Lambda+\rho}$. It is easily shown (using Verma's original proof of the fact that $\dim \operatorname{Hom}_{\mathfrak{g}}(\overline{M}^\Lambda, \overline{M}^\mu) < 1$, $\Lambda, \mu \in \mathfrak{h}$) that each $\overline{M}^{-\Lambda}$ contains a unique $\overline{M}^{-\Lambda'}$ such that $\overline{M}^{-\Lambda'}$ is irreducible.

Using Lemma 1 of Shapavolov [4] it is easily shown that $\overline{M}^{-\Lambda'} \equiv U(\mathfrak{g})n \cdot (1 \otimes 1)$ with $n \in U(\mathfrak{n})$ and $\psi({}^tn) = 1$. (Here ${}^tX = -X$, $X \in \mathfrak{g}$, ${}^t1 = 1$ and ${}^t(g_1g_2) = {}^tg_2{}^tg_1$, $g_i \in U(\mathfrak{g})$.) We note that this is the first time we have used the assumption that ψ is generic.

Since $(\overline{M}^{-\Lambda})^* = \operatorname{Hom}_{U(\bar{\mathfrak{b}})}(U(\mathfrak{g}), \mathbf{C}_{\Lambda-\rho})$, the above observations imply that for

each $\Lambda \in \mathfrak{h}^*$ there exists a unique surjective $\mathfrak{g}$-module homomorphism

$$T_{\Lambda',\Lambda}\colon \operatorname{Hom}_{U(\bar{\mathfrak{b}})}(U(\mathfrak{g}), \mathbf{C}_{\Lambda-\rho}) \to \operatorname{Hom}_{U(\bar{\mathfrak{b}})}(U(\mathfrak{g}), \mathbf{C}_{\Lambda'-\rho})$$

with $T_{\Lambda',\Lambda}(f)(g) = f(n_{\Lambda',\Lambda}\, g)$ and $\psi(n_{\Lambda',\Lambda}) = 1$. In particular, we have

(1) $T_{\Lambda',\Lambda}\omega_\Lambda = \omega_{\Lambda'}$.

We also note that

(2) If $f \in W_\psi (= U(g)\omega)$ then $T_{\Lambda',\Lambda}B_\Lambda(f) = B_{\Lambda'}(f)$.

Indeed,

$$\begin{aligned} T_{\Lambda',\Lambda}B_\Lambda(g\omega) &= T_{\Lambda',\Lambda}g \cdot B_\Lambda(\omega) = g \cdot T_{\Lambda',\Lambda}B_\Lambda(\omega) \\ &= g \cdot T_{\Lambda',\Lambda}\omega_\Lambda = g \cdot \omega_{\Lambda'} = B_{\Lambda'}(g \cdot \omega). \end{aligned}$$

We therefore have

(3) If $f \in \operatorname{Wh}(W_\psi) = W_\psi \cap V_{\psi,0}$ then $(\Lambda - \rho)(f(1)) = (\Lambda' - \rho)(f(1))$ for all $\Lambda \in \mathfrak{h}^*$.

Indeed, $f =_{f(1)} \omega$. $B_\Lambda(F) = (\Lambda - \rho)(f(1))\omega_\Lambda$. But then

$$\begin{aligned} (\Lambda' - \rho)(f(1))B_{\Lambda'}(f) &= T_{\Lambda',\Lambda}(B_\Lambda(f)) \\ &= T_{\Lambda',\Lambda}((\Lambda - \rho)(f(1)))\omega_\Lambda = (\Lambda - \rho)(f(1))\omega_{\Lambda'}. \end{aligned}$$

Let s_0 be the unique element of the Weyl group, W of $(\mathfrak{g}, \mathfrak{h})$, such that $s_0\Delta^+ = -\Delta^+$. If Λ is dominant integral and regular and if $s \in W$ then $(s\Lambda)' = s_0\Lambda$. (3) now implies that if $f \in \operatorname{Wh}(W_\psi)$ then

(4) $(s\Lambda - \rho)(f(1)) = (s_0\Lambda - \rho)(f(1))$ for all $\Lambda \in \mathfrak{h}^*$ dominant integral and regular and all $s \in W$.

Since the integral regular elements of $\mathfrak{h}^*$ are Zariski-dense in $\mathfrak{h}^*$ we have

(5) $(s\Lambda - \rho)(f(1)) = (\Lambda - \rho)(f(1))$ for all $s \in W$ and $\Lambda \in \mathfrak{h}^*$.

Let η: $U(h) \to U(h)$ be the isomorphism defined by $\eta(1) = 1$ and $\eta(H) = H - \rho(H) \cdot 1$, $H \in \mathfrak{h}$. Let $U(\mathfrak{h})^W$ be the algebra of W-invariants in $U(\mathfrak{h})$. Then Harish-Chandra has shown that $\eta \circ \xi$: $\mathfrak{z}(\mathfrak{g}) \to U(h)^W$ is a bijective isomorphism (cf. [1]).

But then if $f \in \operatorname{Wh}(W_\psi)$, $\eta(f(1)) = \eta(\xi(Z))$ for some $Z \in \mathfrak{z}(\mathfrak{g})$. Hence $f(1) = \xi(Z)$ for some $Z \in \mathfrak{z}(\mathfrak{g})$.

Lemma 2.2 now implies that $f = Z \cdot \omega$. Lemma 2.1 now implies Theorem 2.3.

3. The vanishing of $H^1_\psi(\mathfrak{n}; M)$ and its implications. Let $\mathfrak{a}$ be a Lie algebra over $\mathbf{C}$. Let ψ: $\mathfrak{a} \to \mathbf{C}$ be a Lie algebra homomorphism. If M is an $\mathfrak{a}$-module we define $H^i_\psi(\mathfrak{a}, M)$ to be the Lie algebra cohomology of $\mathfrak{a}$ with respect to the action $X \cdot m = (X - \psi(X))m$, $X \in \mathfrak{a}$, $m \in M$.

LEMMA 3.1. *Let $\mathfrak{b}$ be the Lie algebra over $\mathbf{C}$ with basis $H_1, \ldots, H_l, X_1, \ldots, X_l$ and commutation relations $[H_i, X_i] = \delta_{ij}X_j$, $[X_i, X_j] = [H_i, H_j] = 0$, $1 \leq i, j \leq l$. Let M be a module for $\mathfrak{b}$ such that $X_i - 1$ is nilpotent for each $1 < i < l$.*

Set $\mathfrak{n} = \sum_{i=1}^l \mathbf{C}X_i$ and define ψ: $\mathfrak{n} \to \mathbf{C}$ by $\psi(X_i) = 1$, $1 < i < l$. Then $H^1_\psi(\mathfrak{n}, M) = 0$.

PROOF. We first look at the case $l = 1$. Set $X_1 = X$, $H_1 = H$.

(1) X is bijective on M.

Indeed, if $m \in M$, $(X - 1)^k m = 0$. Hence $(-1)^k m \in XM$. If $Xm = 0$ then $(X - 1)m = -m$. Hence $0 = (X - 1)^k m = (-1)^k m$.

Let T: $M \to M$ be the inverse transformation to X: $M \to M$. Then $[X, TH] = -TX = -I$.

(2) $(X - I)^j THm = -j(X - I)^{j-1}m + TH(X - I)^j m$.

This is proven by the obvious induction. Set $M_j = \operatorname{Ker}(X - I)^j$. Then (2) implies

(3) $(X - I)TH + (j + 1)I$: $M_{j+1} \to M_j$.

Since $M_0 = 0$, (3) implies

(4) $(X - I)M = M$.

But (4) is just the statement of the lemma for $l = 1$.

We now prove the lemma by induction on l. We assume the result for $l - 1$. Let f: $\mathfrak{n} \to M$ be a 1-cocycle. That is, $0 = f[X_i, X_j] = (X_i - 1)f(X_j) - (X_j - 1)f(X_i)$.

The inductive hypothesis implies that we can assume $f(X_i) = 0$, $1 \leqslant i \leqslant l - 1$. Hence $0 = f(X_i, X_l) = (X_i - 1)f(X_l)$, $1 \leqslant i \leqslant l - 1$. But then $f(X_l) \in M^0 = \bigcap_{1 \leqslant i \leqslant l-1} \operatorname{Ker}(X_i - 1)$. Clearly, M^0 is invariant under H_l and X_l. Hence there is $v \in M^0$ so that $f(X_l) = (X_l - 1)v$. But $(X_i - 1)v = 0$ for all $1 \leqslant i \leqslant l - 1$. Hence $f(X) = (X - \psi(X))v$ for $X \in \mathfrak{n}$. This proves the lemma.

Let $\mathfrak{p}_n$ denote the semidirect product $\mathfrak{gl}_n(\mathbf{C}) \oplus \mathbf{C}^n$ with $[X, v] = Xv$, $v \in \mathbf{C}^n$, $X \in \mathfrak{gl}_n(\mathbf{C})$ and $[\mathbf{C}^n, \mathbf{C}^n] = 0$.

We denote by $\mathfrak{n}$ the Lie subalgebra $\mathfrak{n}_n \oplus \mathbf{C}^n$ where $\mathfrak{n}_n$ is the space of upper triangular matrices in $\mathfrak{gl}_n(\mathbf{C})$ with zeros on the main diagonal. We identify $\mathfrak{p}_n$ with the maximal parabolic subalgebra of $\mathfrak{sl}_{n+1}(\mathbf{C})$ given by

$$\left[\begin{array}{c|c} \mathfrak{gl}_n(\mathbf{C}) & \mathbf{C}^n \\ \hline 0 & * \end{array}\right].$$

Let $\mathfrak{h} \subset \mathfrak{sl}_{n+1}(\mathbf{C})$ be the diagonal matrices in $\mathfrak{sl}_{n+1}(\mathbf{C})$. Then $\mathfrak{h} \oplus \mathfrak{n}$ is a Borel subalgebra of $\mathfrak{h}$. Let Δ be the root system of $(\mathfrak{sl}(n + 1, \mathbf{C}), \mathfrak{h})$ and let Δ^+ be the simple system of roots corresponding to $\mathfrak{n}$. Let ψ: $\mathfrak{n} \to \mathbf{C}$ be a Lie algebra homomorphism such that $\psi|\mathfrak{n}_\alpha \neq 0$, for α a simple root in Δ^+.

If M is a $\mathfrak{p}_n$-module we say that M is a Whittaker module if $X - \psi(X)$ is nilpotent for $X \in \mathfrak{n}$. (Note that no finite generation is assumed.) If M is a module for $\mathfrak{n}$ we denote by $H^i_\psi(\mathfrak{n}, M)$ the Lie algebra cohomology of $\mathfrak{n}$ with respect to M relative to the action $X \cdot m = (X - \psi(X))m$.

THEOREM 3.2. *Let M be a Whittaker module for $\mathfrak{p}_n$. Then $H^1_\psi(\mathfrak{n}, M) = 0$.*

PROOF. By induction on n. If $n = 1$ this is just Lemma 3.1. Assume that the result is true for $1 \leqslant n \leqslant j - 1$ and that $n = j$. Let f: $\mathfrak{n} \to M$ be a 1-cocycle, that is

$$f([X, Y]) = (X - \psi(X))f(Y) - (Y - \psi(Y))f(X), \qquad X, Y \in \mathfrak{n}.$$

We must show that there exists $v \in M$ so that $f(X) = (X - \psi(X))v$, $X \in \mathfrak{n}$.

Let $v_1, \ldots, v_n$ be a basis of $\mathbf{C}^n$ so that $\psi(v_i) = 1$, $i = 1, \ldots, n$. (This is clearly possible.) Let $H_1, \ldots, H_l \in \mathfrak{gl}_n(\mathbf{C})$ be such that $[H_i v_j] = \delta_{ij} v_j$ and $[H_i, H_j] = 0$. Again, it is clear that the H_i exist. Lemma 3.1 now implies that there exists $u \in M$ so that

$$f(v_i) = (v_i - 1)u, \qquad 1 \leqslant i \leqslant n.$$

Thus replacing f with $f - g$, $g(X) = (X - \psi(X))u$, $X \in \mathfrak{n}$, we may assume $f(\mathbf{C}^n) = 0$.

Let $e_1, \ldots, e_n$ be the standard basis of $\mathbf{C}^n$. Then (after possibly changing e_n by a scalar multiple), $\psi(e_i) = 0$, $i < n - 1$, and $\psi(e_n) = 1$.

If $X \in \mathfrak{n}_n$ and if $v \in \mathbf{C}^n$ then $0 = f([v, X]) - (v - \psi(v))f(X)$. Thus

$$(v - \psi(v))f(X) = 0 \quad \text{for } v \in \mathbf{C}^n.$$

Set $M^0 = \{m \in M \mid v \cdot m = \psi(v)m, v \in \mathbf{C}^n\}$.

Let

$$\mathfrak{p}_{n-1} = \begin{array}{|c|c|} \hline \mathfrak{gl}_{n-1}(\mathbf{C}) & \mathbf{C}^{n-1} \\ \hline 0 & 0 \\ \hline \end{array}$$

in $\mathfrak{gl}_n(\mathbf{C})$. Then $\mathfrak{p}_{n-1} \supset \mathfrak{n}_n$ and $\mathfrak{p}_{n-1} \cdot M^0 \subset M^0$. Since $f\colon \mathfrak{n}_n \to M^0$ the inductive hypothesis implies that there exists $w \in M^0$ such that $f(X) = (X - \psi(X))w$, $X \in \mathfrak{n}_n$. But then it is clear that $f(X) = (X - \psi(X))w$, $X \in \mathfrak{n}$. Q.E.D.

We have given the above proof of Theorem 3.2 since it is quite elementary. However, using the Hochschild-Serre spectral sequence, it is easily proven that if M is a Whittaker module for $\mathfrak{p}_n$ then $H^i_\psi(\mathfrak{n}, M) = 0$ for $i > 0$. We now sketch this result. We first prove that under the hypothesis of Lemma 3.1, $H^i_\psi(\mathfrak{n}, M) = 0$ for $i > 0$. This is done by induction on l. For $l = 1$ this is the statement of Lemma 3.1. If true for $l - 1$ and we are in the case of l then $\mathfrak{n} \supset \mathbf{C}X_l$ as a normal subalgebra and the Hochschild-Serre spectral sequence has E_2 term

$$H^p_\psi(\mathfrak{n}/\mathbf{C}X_l, H^q_\psi(\mathbf{C}X_l, M)) \Rightarrow H^{p+q}_\psi(\mathfrak{n}, M).$$

$H^q_\psi(\mathbf{C}X_l, M) = 0$ for $q > 0$ and $H^0_\psi(\mathbf{C}X_l, M)$ is a Whittaker module for $\sum_{i=1}^{l-1} \mathbf{C}X_i \oplus \sum_{i=1}^{l-1} \mathbf{C}H_i$. Thus $E_2^{p,q} = 0$ if $p > 0$ or $q > 0$. The spectral sequence thus degenerates at the E_2 term and yields the desired result. We now return to the hypotheses of Theorem 3.1. Using the same induction as in the proof of Theorem 3.1 and the above argument using the Hochschild-Serre spectral sequence we have

THEOREM 3.2′. *If M is a Whittaker module for $\mathfrak{p}_n$ then $H^i_\psi(\mathfrak{n}, M) = 0$ for $i > 0$.*

Let $\mathbf{C}_\psi$ be the $\mathfrak{n}$-module $\mathbf{C}$ with action ψ.

LEMMA 3.3. *$U(\mathfrak{p}_n) \otimes_{U(\mathfrak{n})} \mathbf{C}_\psi$ is irreducible as a $\mathfrak{p}_n$-module.*

PROOF. By induction on n. If $n = 0$ the result is clear. We assume the result for $n - 1$ and prove it for n. Let $\delta = \psi|_{\mathbf{C}^n}$, ($\mathfrak{p}_n = \mathfrak{gl}_n(\mathbf{C}) \oplus \mathbf{C}^n$). Let

$$\mathrm{St}(\delta, \mathfrak{p}_n) = \{ Y \in \mathfrak{p}_n \mid \delta([Y, X]) = 0, X \in \mathbf{C}^n \}.$$

Then

$$\mathrm{St}(\delta, \mathfrak{p}_n) = \mathfrak{p}_{n-1} \oplus \mathbf{C}^n.$$

Set $N = U(\mathrm{St}(\delta, \mathfrak{p}_n)) \otimes_{U(\mathfrak{n})} \mathbf{C}_\psi$. Then as a $\mathfrak{p}_{n-1}$-module $N = U(\mathfrak{p}_{n-1}) \otimes_{U(\mathfrak{n}\cap\mathfrak{p}_{n-1})} \mathbf{C}_\psi$. Thus N is irreducible by the inductive hypothesis. But

$$U(\mathfrak{p}_n) \otimes_{U(\mathfrak{n})} \mathbf{C}_\psi = U(\mathfrak{p}_n) \otimes_{U(\mathrm{St}(\delta,\mathfrak{p}_n))} N,$$

and it is clear that if $X \in \mathbf{C}^n$ then X acts on N by $\psi(X)$. Hence Dixmier [**1**, 5.3.6, p. 180] implies the result.

Set $\mathfrak{n} = \mathfrak{n}^n = \mathfrak{n}_n \oplus \mathbf{C}^n$, $\psi = \psi_n$. We now look at the case when $\mathfrak{p} = \times_{i=1}^{n} \mathfrak{p}_{n_i}$, $\mathfrak{n} = \times_{i=1}^{r} \mathfrak{n}^{n_i}$ and $\psi = \times_{i=1}^{r} \psi_{n_i}$. A Whittaker module for $\mathfrak{p}$ will mean (as before) a $\mathfrak{p}$-module, M, such that if $X \in \mathfrak{n}$, $X - \psi(X)$ is nilpotent on M. $H^i_\psi(\mathfrak{n}, M)$ is the ith Lie algebra cohomology of $\mathfrak{n}$ with respect to the action $X \cdot m = (X - \psi(X))m$ on M.

THEOREM 3.4. *If M is a Whittaker module for $\mathfrak{p}$ then $H^1_\psi(\mathfrak{n}, M) = 0$.*

PROOF. By induction on r. If $r = 1$ this is just Theorem 3.2. Assume that the theorem is true for $r - 1$. Let $f\colon \mathfrak{n} \to M$ be a cocycle. Then the case $r = 1$ implies that we may assume $f(\mathfrak{n}^{n_r}) = 0$. If $X \in \bigoplus_{i=1}^{r-1} \mathfrak{n}^{n_i}$ and $Y \in \mathfrak{n}^{n_r}$ then $0 = f([Y, X]) = (Y - \psi(Y))f(X)$. Thus $f\colon \mathfrak{n} \to M^0$ where

$$M^0 = \{m \in M \mid (Y - \psi(Y))m = 0,\ Y \in \mathfrak{n}^{n_r}\}.$$

Now argue as in the inductive step in Lemma 3.1.

Throughout the rest of this section we take $\mathfrak{g} = \times_{i=1}^{r} \mathfrak{sl}_{n_i}(\mathbf{C})$. $\mathfrak{p} = \times_{i=1}^{r} \mathfrak{p}_{n_i}$, $\mathfrak{n} = \times_{i=1}^{r} \mathfrak{n}^{n_i}$, ψ as above, $b = \mathfrak{h} \oplus \mathfrak{n}$, $\bar{\mathfrak{b}} = \mathfrak{h} \oplus \bar{\mathfrak{n}}$ as in §2. A Whittaker module for $\mathfrak{g}$ is a $\mathfrak{g}$ module, M, such that $X - \psi(X)$ is nilpotent for $X \in \mathfrak{n}$. Arguing as in the proof of Theorem 3.2′ and using the inductive argument of the proof of Theorem 3.4 we have

THEOREM 3.4′. *If M is a Whittaker module for $\mathfrak{p}$ then $H^i_\psi(\mathfrak{n}, M) = 0$ for $i > 0$.*

THEOREM 3.5. *Let $0 \neq M$ be an irreducible Whittaker module for $\mathfrak{g}$. Then*

(1) (Compare Kostant [**3**, Theorem 3.3].) *M has infinitesimal character χ and $M \equiv U(\mathfrak{g}) \otimes_{\mathfrak{z}(\mathfrak{g})\otimes U(\mathfrak{n})} \mathbf{C}_{\chi,\psi} = M_{\chi,\psi}$. (Here $\mathbf{C}_{\chi,\psi}$ is the $\mathfrak{z}(\mathfrak{g}) \otimes U(\mathfrak{n})$-module $\mathbf{C}$ with $\mathfrak{z}(\mathfrak{g})$ acting by χ, $U(\mathfrak{n})$ acting by ψ.) Furthermore, $M_{\chi,\psi}$ is irreducible for each $\chi\colon \mathfrak{z}(\mathfrak{g}) \to \mathbf{C}$ a homomorphism.*

(2) *M is irreducible as a $\mathfrak{p}$-module and as a $\mathfrak{p}$-module $M \equiv U(\mathfrak{p}) \otimes_{U(\mathfrak{n})} \mathbf{C}_\psi$.*

PROOF. By Dixmier's Schur's lemma (cf. [**1**, 2.6.5]) M has infinitesimal character χ.

There is thus a surjective $\mathfrak{g}$-module homomorphism, $M_{\chi,\psi} \xrightarrow{\alpha} M$. Also, we have a natural surjection $\beta\colon M_\psi \to M_{\chi,\psi}$, $\beta(1 \otimes 1) = 1 \otimes 1$. Let $Q = \operatorname{Ker} \beta$. Then the exact sequence

$$0 \to Q \to M_\psi \to M_{\chi,\psi} \to 0$$

induces the exact sequence

$$0 \to H^0_\psi(\mathfrak{n}, Q) \to H^0_\psi(\mathfrak{n}, M) \to H^0_\psi(\mathfrak{n}, M_{\chi,\psi}) \to H^1_\psi(\mathfrak{n}, Q).$$

Now $H^0_\psi(\mathfrak{n}, R) = \mathrm{Wh}(R)$ for R an $\mathfrak{n}$-module. Also $H^1_\psi(\mathfrak{n}, Q) = 0$ by Theorem 3.4.

Thus

$$0 \to \mathrm{Wh}(Q) \to \mathrm{Wh}(M_\psi) \to \mathrm{Wh}(M_{\chi,\psi}) \to 0$$

is exact.

But $\mathrm{Wh}(M_\psi) = \mathfrak{z}(\mathfrak{g}) \otimes 1$ (Theorem 2.3). Thus $\dim \alpha(\mathrm{Wh}(M_\psi)) < 1$. Since $\alpha \neq 0$, $\dim(\mathrm{Wh}(M_\psi)) = 1$. Hence $\dim \mathrm{Wh}(M_{\chi,\psi}) = 1$. This implies $M_{\chi,\psi}$ is irreducible and that α is injective.

We thus may assume $M = M_{\chi,\psi}$. Let $0 \neq P \subset M$ be a $\mathfrak{p}$-submodule of M. Then the exact sequence

$$0 \to P \to M \to M/P \to 0$$

induces (by Theorem 3.4)

$$0 \to \mathrm{Wh}(P) \to \mathrm{Wh}(M) \to \mathrm{Wh}(M/P) \to 0.$$

But $P \neq 0$ and $\dim \mathrm{Wh}(M) = 1$. Thus $\dim \mathrm{Wh}(P) = 1$. Thus $\mathrm{Wh}(M/P) = 0$. This clearly implies $M = P$.

Applying the above and Lemma 3.3 to the case of $\mathfrak{p} = \mathfrak{p}_n$, we find that $\dim \mathrm{Wh}(U(\mathfrak{p}_n) \otimes_{U(\mathfrak{n})} \mathbf{C}_\psi) = 1$. In general $U(\mathfrak{p}) \otimes_{U(\mathfrak{n})} \mathbf{C}_\psi$ is a tensor product

$$\left(U(\mathfrak{p}_{n_1}) \otimes_{U(\mathfrak{n}^{n_1})} \mathbf{C}_\psi\right) \otimes \cdots \otimes \left(U(\mathfrak{p}_{n_r}) \otimes_{U(\mathfrak{n}^{n_r})} \mathbf{C}_\psi\right).$$

It is thus easy to see that

$$\dim \mathrm{Wh}\left(U(\mathfrak{p}) \otimes_{U(\mathfrak{n})} \mathbf{C}_\psi\right) = 1.$$

But then $U(\mathfrak{p}) \otimes_{U(\mathfrak{n})} \mathbf{C}_\psi$ is irreducible. Since M is clearly a quotient of $U(\mathfrak{p}) \otimes_{U(\mathfrak{n})} \mathbf{C}_\psi$ the last part of (2) follows.

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