GROUPS OF FINITE WEIGHT

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ABSTRACT. If $N$ is a group and $E$ is a group of operators on $N$ then write $d_E(N)$ for the minimum number of elements needed to generate $N$ as an $E$-group. It is shown that if $N$ is a normal subgroup of $E$ and $E$ acts on $N$ by conjugation, then $d_E(N) = d_E(N/N')$ if $d_E(N)$ is finite and there does not exist an infinite descending series of $E$-normal subgroups $N' = C_0 > C_1 > \cdots$ with each $C_i/C_{i+1}$ perfect. Both these conditions are, in general, necessary.

If $N$ is a group and $E$ is a group of operators $N$ then we shall write $d_E(N)$ for the minimum number of elements needed to generate $N$ as an $E$-group. We shall call this number the $E$-weight of $N$ and use the convention $d_E(N) = 1$ if $N$ is the trivial group. The main result of this note is to prove the following result referred to by K. W. Gruenberg in [2, p. 74].

**Theorem.** If $N$ is a normal subgroup of a group $E$ and $E$ acts on $N$ by conjugation, then $d_E(N) = d_E(N/N')$ provided $d_E(N)$ is finite and $N$ has the following property.

(•) There does not exist an infinite descending series of $E$-subgroups $N' = C_0 > C_1 > \cdots$ with each $C_i/C_{i+1}$ perfect.

We have used $N'$ to denote the commutator subgroup of $N$. A group $G$ is perfect if $G = G'$. Note that if the lattice of normal subgroups of $E$ contained in $N'$ satisfies the minimum condition then the property (•) is trivially satisfied. Thus as a corollary we obtain the following result of P. Kutzko [5].

If $G$ is a group such that $d_G(G)$ is finite and the lattice of normal subgroups of $G$ which are contained in $G'$ satisfies the minimum condition then $d_G(G) = d_G(G/G')$.

It is important to point out that the above result of Kutzko was obtained by R. Baer in [1]. The purpose of this note is to supply a proof of a result mentioned in the literature.

**Proof of the theorem.** Suppose the result is false. Let $C_0 = N'$. Let $d_E(N/C_0) = k$. Then there exist $a_1, \ldots, a_k$ in $N$ such that $A_0 = \langle a_1, \ldots, a_k \rangle_E$ satisfies $A_0C_0 = N$. If $A_0 > C_0$ then $A_0 = N$ and we have the required contradiction. Let $B_0 = A_0 \cap C_0$ so that $B_0 < C_0$. Note that since $C_0 < N'$, $C_0/B_0$ is perfect. Also since $d_E(N)$ is finite, so is $d_E(N/B_0)$. Thus there exists a normal subgroup $C_1$ of $E$ such that $B_0 < C_1 < C_0$ and $d_E(C_0/C_1) = 1$. Now $C_0/C_1$ is perfect and $C_0 = \langle C_1, x \rangle_E$ for some $x$ in $C_0$.

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Let $A_1 = \langle a_1, x, a_2, \ldots, a_k \rangle^E$. Since $[a_1, C_0] \leq C_0 \cap A_0 = B_0 < C_1$, $C_1[a_1, x, C_0]^E = C_1[x, C_0]^E$ and $C_0/C_1[x, C_0]^E$ is cyclic. But $C_0/C_1$ is perfect. Hence $C_1[a_1, x, C_0]^E = C_0$ and $A_1C_1 = N$. Note that $d_E(N/C_1) = k$. Let $B_1 = A_1 \cap C_1$ and repeat the above argument with $A_1$, $B_1$, $C_1$ replacing $A_0$, $B_0$, $C_0$ respectively. This process yields a sequence of $N' = C_0 > C_1 > \cdots$ where $C_i$ are normal in $E$ and each $C_i/C_{i+1}$ is perfect, contradicting property $(*)$. This completes the proof.

**Remark 1.** The condition that $d_E(N)$ be finite is, in general, necessary in the above theorem. Heineken and Wilson’s example in [3] shows this. They constructed a countable, locally solvable, perfect group $G$ satisfying the minimal condition on normal subgroups and at the same time $G$ is isomorphic to all its nontrivial homomorphic images. Since $G$ is perfect, $d_G(G/G') = 1$. By the above theorem $d_G(G) = 1$ or $\infty$. If $G = \langle g^G \rangle$ for some $g$ in $G$, then look at $G/M$ where $M$ is a maximal normal subgroup of $G$ subject to not containing $g$. Then every normal subgroup of $G/M$ contains $Mg$. Thus $G/M$ is simple. But $G/M$ is locally solvable and, as is well known, a locally solvable simple group is trivial. Thus $d_G(G) = \infty$.

**Remark 2.** The condition $(*)$ in the theorem is, in general, necessary. M. A. Kervaire has shown in [4] that the free product $F$ of $G$ and $C_\infty$ where $G = \langle \alpha, \beta; \alpha^2 = \beta^2 = (\alpha^{-1}\beta)^5 \rangle$ has the property that $F/F' \cong C_\infty$, the infinite cyclic group, so that $d_F(F/F') = 1$. But $d_F(F) > 1$.

**References**


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