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GROUPS OF FINITE WEIGHT

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ABSTRACT. If N is a group and E is a group of operators on N then write $d_E(N)$ for the minimum number of elements needed to generate N as an E-group. It is shown that if N is a normal subgroup of E and E acts on N by conjugation, then $d_E(N) = d_E(N/N')$ if $d_E(N)$ is finite and there does not exist an infinite descending series of E-normal subgroups $N' = C_0 > C_1 > \cdots$ with each C_i/C_{i+1} perfect. Both these conditions are, in general, necessary.

If N is a group and E is a group of operators N then we shall write $d_E(N)$ for the minimum number of elements needed to generate N as an E-group. We shall call this number the E-weight of N and use the convention $d_E(N) = 1$ if N is the trivial group. The main result of this note is to prove the following result referred to by K. W. Gruenberg in [**2**, p. 74].

THEOREM. *If N is a normal subgroup of a group E and E acts on N by conjugation, then $d_E(N) = d_E(N/N')$ provided $d_E(N)$ is finite and N has the following property.*

($*$) *There does not exist an infinite descending series of E-subgroups $N' = C_0 > C_1 > \cdots$ with each C_i/C_{i+1} perfect.*

We have used N' to denote the commutator subgroup of N. A group G is perfect if $G = G'$. Note that if the lattice of normal subgroups of E contaned in N' satisfies the minimum condition then the property ($*$) is trivially satisfied. Thus as a corollary we obtain the following result of P. Kutzko [**5**].

If G is a group such that $d_G(G)$ is finite and the lattice of normal subgroups of G which are contained in G' satisfies the minimum condition then $d_G(G) = d_G(G/G')$.

It is important to point out that the above result of Kutzko was obtained by R. Baer in [**1**]. The purpose of this note is to supply a proof of a result mentioned in the literature.

PROOF OF THE THEOREM. Suppose the result is false. Let $C_0 = N'$. Let $d_E(N/C_0) = k$. Then there exist $a_1, \ldots, a_k$ in N such that $A_0 = \langle a_1, \ldots, a_k\rangle^E$ satisfies $A_0C_0 = N$. If $A_0 \geqslant C_0$ then $A_0 = N$ and we have the required contradiction. Let $B_0 = A_0 \cap C_0$ so that $B_0 < C_0$. Note that since $C_0 \leqslant N'$, C_0/B_0 is perfect. Also since $d_E(N)$ is finite, so is $d_E(N/B_0)$. Thus there exists a normal subgroup C_1 of E such that $B_0 \leqslant C_1 < C_0$ and $d_E(C_0/C_1) = 1$. Now C_0/C_1 is perfect and $C_0 = \langle C_1, x\rangle^E$ for some x in C_0.

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Let $A_1 = \langle a_1x, a_2, \ldots, a_k\rangle^E$. Since $[a_1, C_0] < C_0 \cap A_0 = B_0 < C_1$, $C_1[a_1x, C_0]^E = C_1[x, C_0]^E$ and $C_0/C_1[x, C_0]^E$ is cyclic. But C_0/C_1 is perfect. Hence $C_1[a_1x, C_0]^E = C_0$ and $A_1C_1 = N$. Note that $d_E(N/C_1) = k$. Let $B_1 = A_1 \cap C_1$ and repeat the above argument with A_1, B_1, C_1 replacing A_0, B_0, C_0 respectively. This process yields a sequence of $N' = C_0 > C_1 > \cdots$ where C_i are normal in E and each C_i/C_{i+1} is perfect, contradicting property (*). This completes the proof.

REMARK 1. The condition that $d_E(N)$ be finite is, in general, necessary in the above theorem. Heineken and Wilson's example in [3] shows this. They constructed a countable, locally solvable, perfect group G satisfying the minimal condition on normal subgroups and at the same time G is isomorphic to all its nontrivial homomorphic images. Since G is perfect, $d_G(G/G') = 1$. By the above theorem $d_G(G) = 1$ or ∞. If $G = \langle g^G\rangle$ for some g in G, then look at G/M where M is a maximal normal subgroup of G subject to not containing g. Then every normal subgroup of G/M contains Mg. Thus G/M is simple. But G/M is locally solvable and, as is well known, a locally solvable simple group is trivial. Thus $d_G(G) = \infty$.

REMARK 2. The condition (*) in the theorem is, in general, necessary. M. A. Kervaire has shown in [4] that the free product F of G and C_∞ where $G = \langle \alpha, \beta; \alpha^2 = \beta^2 = (\alpha^{-1}\beta)^5\rangle$ has the property that $F/F' \cong C_\infty$, the infinite cyclic group, so that $d_F(F/F') = 1$. But $d_F(F) > 1$.

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