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FIXED SUBALGEBRA OF A COMMUTATIVE FROBENIUS ALGEBRA

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ABSTRACT. Let B be a finite-dimensional commutative algebra generated by a single element, and let $A = B \otimes B$. We prove that the fixed subalgebra of A under the involution $b_1 \otimes b_2 \mapsto b_2 \otimes b_1$ is Frobenius if and only if either the characteristic of B is different from 2 or B is separable.

1. Introduction. Let A be a (finite-dimensional) Frobenius algebra over a field and G a finite group of automorphisms of A. As is shown by J. L. Pascaud and J. Valette [3], it does not necessarily follow then that the fixed subalgebra A^G under G is Frobenius; indeed, they give an example of a local commutative Frobenius A of any characteristic different from 2 and a group G of order 2 such that A^G is not Frobenius. (Note that for commutative algebras the concepts of "Frobenius" and "quasi-Frobenius" coincide.) In connection with this, N. Jacobson has raised the following question: Suppose B is a commutative Frobenius algebra, or more particularly, an algebra generated by a single element, $A = B \otimes B$, and G is the group of order 2 generated by the automorphism σ: $b_1 \otimes b_2 \mapsto b_2 \otimes b_1$ of A. Is then A^G Frobenius too? In the present paper the question is answered affirmatively in the case where B is singly generated and the characteristic of B is different from 2.

2. Lemmas. Let B be a finite-dimensional commutative algebra over a field K, $A = B \otimes_K B$ and $G = \{1, \sigma\}$, where σ is the automorphism as above. Suppose that $w_1, w_2, \ldots, w_n$ are a basis of B over K. Then the n^2 elements $w_i \otimes w_j$ $(1 \leq i, j \leq n)$ form a basis of A. If we observe that $(w_i \otimes w_j)^\sigma = w_j \otimes w_i$ for all i, j, we can easily derive the following

LEMMA 1. *Let $w_1, w_2, \ldots, w_n$ be a basis of B over K. Then the n elements $w_i \otimes w_i$ $(1 \leq i \leq n)$ and $n(n-1)/2$ elements $w_i \otimes w_j + w_j \otimes w_i$ $(1 \leq i < j \leq n)$ together form a basis of A^G over K.*

Let L be an extension field of K. Then $B \otimes_K L$ is considered a commutative algebra over L, and we have $A \otimes_K L = (B \otimes_K L) \otimes_L (B \otimes_K L)$. The automorphism σ of A can also be regarded as an automorphism of $A \otimes_K L$ in the natural manner. Moreover, if $w_1, w_2, \ldots, w_n$ are a basis of B over K then it is also a basis

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of $B \otimes_K L$ over L, and therefore by applying Lemma 1 to $B \otimes_K L$ and to this basis we know that the $n + n(n-1)/2$ $(= n(n+1)/2)$ elements given in Lemma 1 together form a basis of $(A \otimes_K L)^G$ over L. Thus we have

LEMMA 2. *Let L be an extension field of K. Then $(A \otimes_K L)^G = A^G \otimes_K L$.*

Let $B = B_1 \oplus B_2 \oplus \cdots \oplus B_r$ be a decomposition of B into a direct sum of orthogonal subalgebras. Put $A_{\alpha,\beta} = B_\alpha \otimes_K B_\beta$ for every α, β $(= 1, 2, \ldots, r)$. Then $A_{\alpha,\beta}$'s are orthogonal subalgebras of A such that $A = \sum \oplus A_{\alpha,\beta}$. Each $A_{\alpha,\beta}$ clearly satisfies $A_{\alpha,\beta}^\sigma = A_{\beta,\alpha}$, and in particular every $A_{\alpha,\alpha}$ is a G-invariant subalgebra. Let (α, β) be a pair such that $1 \leqslant \alpha < \beta \leqslant r$, and put $\tilde{A}_{\alpha,\beta} = A_{\alpha,\beta} \oplus A_{\beta,\alpha}$. Then $\tilde{A}_{\alpha,\beta}$ is also a G-invariant subalgebra, and we have $\tilde{A}_{\alpha,\beta}^G = \{a + a^\sigma | a \in A_{\alpha,\beta}\}$, since $A_{\alpha,\beta}$ and $A_{\beta,\alpha}$ are orthogonal; thus we know that, by associating $a \in A_{\alpha,\beta}$ with $a + a^\sigma$, $\tilde{A}_{\alpha,\beta}^G$ is isomorphic to $A_{\alpha,\beta}$ as algebras. We have therefore the following

LEMMA 3. *Let B be a direct sum of orthogonal subalgebras B_α, $1 \leqslant \alpha \leqslant r$. Let $A_{\alpha,\beta} = B_\alpha \otimes_K B_\beta$ for every α, β, and let $\tilde{A}_{\alpha,\beta} = A_{\alpha,\beta} \oplus A_{\beta,\alpha}$ for every α, β such that $1 \leqslant \alpha < \beta \leqslant r$. Then the $r(r+1)/2$ algebras $A_{\alpha,\alpha}$ and $\tilde{A}_{\alpha,\beta}$ are orthogonal G-invariant subalgebras of $A = B \otimes_K B$, and A is a direct sum of these subalgebras; moreover, every algebra $\tilde{A}_{\alpha,\beta}^G$ is isomorphic to $A_{\alpha,\beta}$.*

LEMMA 4. *Let B be a commutative Frobenius algebra, and let $A = B \otimes_K B$. Then A^G is Frobenius if and only if $A_{\alpha,\alpha}^G$ is Frobenius for every α.*

PROOF. From Lemma 3 it follows that A^G is the direct sum of orthogonal subalgebras $A_{\alpha,\alpha}^G$ and $\tilde{A}_{\alpha,\beta}^G$, where the latter algebras are isomorphic to $A_{\alpha,\beta}$. Therefore A^G is Frobenius if and only if all $A_{\alpha,\alpha}^G$ and $A_{\alpha,\beta}$ are Frobenius. Since however B is a Frobenius algebra, each direct summand subalgebra B_α and hence $A_{\alpha,\beta} = B_\alpha \otimes_K B_\beta$ are Frobenius algebras too. Therefore, in order that A^G be Frobenius it is (necessary and) sufficient that $A_{\alpha,\alpha}^G$ be Frobenius for all α.

3. Main results.

PROPOSITION 5. *Let B be an algebra generated by a nonzero nilpotent element over a field K. Then A^G is a Frobenius algebra if and only if the characteristic of K is different from 2.*

PROOF. Let $B = K[c]$ with a nilpotent element $c \neq 0$ of B. Let n be the least positive integer such that $c^n = 0$. Then $n \geqslant 2$, and the n elements $1, c, c^2, \ldots, c^{n-1}$ form a basis of B over K. Therefore, by Lemma 1, the n elements $c^i \otimes c^i$ $(0 \leqslant i \leqslant n-1)$ and $n(n-1)/2$ elements $c^i \otimes c^j + c^j \otimes c^i$ $(0 \leqslant i < j \leqslant n-1)$ together form a basis of A^G over K. If we remove $1 \otimes 1$ (the unit element of A^G) out of the basis then we have a subset W. Let N be the K-subspace of A^G generated by W. Then since every element of W is nilpotent, N is contained in the radical of A^G. It is however clear that $A^G = N \oplus K(1 \otimes 1)$, and this, together with the fact that the radical is a proper ideal of A^G, implies that N coincides with the radical and the factor algebra A^G/N is isomorphic to K. Thus, A^G is a local ring and has an up-to-isomorphism unique simple module, which is one-dimensional over K. Moreover, N is the ideal of A^G generated by $c \otimes c$ and

$1 \otimes c + c \otimes 1$, because $c^i \otimes c^i = (c \otimes c)^i$ for $i > 0$,

$$1 \otimes c^2 + c^2 \otimes 1 = (1 \otimes c + c \otimes 1)^2 - 2(c \otimes c),$$

$$1 \otimes c^i + c^i \otimes 1 = (1 \otimes c + c \otimes 1)(1 \otimes c^{i-1} + c^{i-1} \otimes 1) - (c \otimes c)(1 \otimes c^{i-2} + c^{i-2} \otimes 1) \quad \text{for } i > 2$$

and

$$c^i \otimes c^j + c^j \otimes c^i = (c \otimes c)^i(1 \otimes c^{j-i} + c^{j-i} \otimes 1) \quad \text{for } j > i > 0.$$

(More precisely, by using these equalities, it is possible to show without difficulty that every element of W is expressed as a polynomial of $c \otimes c$ and $1 \otimes c + c \otimes 1$ with integral coefficients and without constant term.)

Let now M be the socle of A^G, i.e., the sum of all simple ideals of A^G. As is well known, M is then the annihilator of the radical N in A^G and therefore is the annihilator of the generators $c \otimes c$ and $1 \otimes c + c \otimes 1$ of N in A^G. Let M' be the annihilator of $c \otimes c$ in A^G. If we observe that

$$(c \otimes c)(c^i \otimes c^i) = c^{i+1} \otimes c^{i+1}$$

is a member of W or $= 0$ according as $0 \leqslant i \leqslant n - 2$ or $j = n - 1$ and also

$$(c \otimes c)(c^i \otimes c^j + c^j \otimes c^i) = c^{i+1} \otimes c^{j+1} + c^{j+1} \otimes c^{i+1}$$

is a member of W or $= 0$ according as $0 \leqslant i < j \leqslant n - 2$ or $j = n - 1$, then we know that M' is the K-subspace of A^G generated by $c^{n-1} \otimes c^{n-1}$, $1 \otimes c^{n-1} + c^{n-1} \otimes 1$, $c \otimes c^{n-1} + c^{n-1} \otimes c, \ldots, c^{n-2} \otimes c^{n-1} + c^{n-1} \otimes c^{n-2}$. Next multiply these n elements by $1 \otimes c + c \otimes 1$. Then we have

$$(1 \otimes c + c \otimes 1)(c^{n-1} \otimes c^{n-1}) = 0,$$

$$(1 \otimes c + c \otimes 1)(1 \otimes c^{n-1} + c^{n-1} \otimes 1) = c \otimes c^{n-1} + c^{n-1} \otimes c, \ldots,$$

$$(1 \otimes c + c \otimes 1)(c^{n-3} \otimes c^{n-1} + c^{n-1} \otimes c^{n-3}) = c^{n-2} \otimes c^{n-1} + c^{n-1} \otimes c^{n-2},$$

$$(1 \otimes c + c \otimes 1)(c^{n-2} \otimes c^{n-1} + c^{n-1} \otimes c^{n-2}) = 2(c^{n-1} \otimes c^{n-1}),$$

where the last element is 0 if and only if the characteristic of K is 2. This means that the annihilator of $1 \otimes c + c \otimes 1$ in M', i.e., the annihilator M of N in A^G is $K \cdot (c^{n-1} \otimes c^{n-1})$ or $K \cdot (c^{n-1} \otimes c^{n-1}) \oplus K \cdot (c^{n-2} \otimes c^{n-1} + c^{n-1} \otimes c^{n-2})$ according as the characteristic of K is different from 2 or equal to 2. Now, since A^G is a local ring, A^G is a Frobenius algebra if and only if A^G has a unique simple ideal, or equivalently, its socle M is simple. But since A^G/N is isomorphic to K, this is equivalent to the condition that M is one-dimensional over K. Thus we have that A^G is Frobenius if and only if $M = K \cdot (c^{n-1} \otimes c^{n-1})$, i.e., the characteristic of K is different from 2.

COROLLARY 6. *Let $B = K[a]$, and let the minimal polynomial $f(t)$ of a over K be a power of a linear polynomial in K. Then A^G is a Frobenius algebra if and only if the characteristic of K is different from 2 or $f(t)$ is linear.*

PROOF. Let $f(t) = (t - \lambda)^n$, $\lambda \in K$. If we put $c = a - \lambda$ then $B = K[c]$ and n is the least positive integer such that $c^n = 0$. Thus, if $n > 1$ then $c \neq 0$ and by Proposition 5, A^G is Frobenius if and only if the characteristic of K is different

from 2, while if $n = 1$ then $c = 0$ and clearly $B = K = A = A^G$ and the K-algebra K is trivially a Frobenius algebra.

THEOREM 7. *Let $B = K[a]$ and let $f(t)$ be the minimal polynomial of a over K. Let $A = B \otimes_K B$ and let $G = \{1, \sigma\}$, where σ is the automorphism of A defined by $(b_1 \otimes b_2)^\sigma = b_2 \otimes b_1$ for $b_1, b_2 \in B$. Then A^G is a Frobenius algebra over K if and only if the characteristic of K is different from 2 or $f(t)$ is separable (i.e., $f(t)$ has no multiple root in a splitting field).*

PROOF. (i) Assume first that K is algebraically closed. Then $f(t)$ is decomposed as

$$f(t) = (t - \lambda_1)^{n_1}(t - \lambda_2)^{n_2} \cdots (t - \lambda_r)^{n_r},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_r$ are distinct roots of $f(t)$ in K. As is well known, this decomposition yields a direct decomposition $B = B_1 \oplus B_2 \oplus \cdots \oplus B_r$ into orthogonal subalgebras such that if a_α is the B_α-component of a then $B_\alpha = K[a_\alpha]$ and $(t - \lambda_\alpha)^{n_\alpha}$ is the minimal polynomial of a_α over K for every α. Now, as a homomorphic image of the polynomial ring $K[t]$, B is a principal ideal ring and therefore a Frobenius algebra (cf. Nakayama [**1**, Theorem 16]). From this follows by Lemma 4 that A^G is Frobenius if and only if $A^G_{\alpha,\alpha}$, where $A_{\alpha,\alpha} = B_\alpha \otimes_K B_\alpha$, is Frobenius for every α. On the other hand, according to Corollary 6, each $A^G_{\alpha,\alpha}$ is Frobenius if and only if the characteristic of K is different from 2 or $n_\alpha = 1$. Thus we have that A^G is a Frobenius algebra if and only if either the characteristic of K is different from 2 or $n_1 = n_2 = \cdots = n_r = 1$.

(ii) Suppose that K is not necessarily algebraically closed. Let L be the algebraic closure of K. Then $(A \otimes_K L)^G = A^G \otimes_K L$ by Lemma 2. Therefore it follows by Nakayama [**1**, Theorem 14] or Nakayama and Nesbitt [**2**, Theorem 5] that A^G is Frobenius over K if and only if $(A \otimes_K L)^G$ is Frobenius over L. But if we observe that $A \otimes_K L = (B \otimes_K L) \otimes_L (B \otimes_K L)$, $B \otimes_K L = L[a]$ and the minimal polynomial of a over L is the same as $f(t)$, we can conclude by the above case (i) that the latter condition is equivalent to the condition that the characteristic of L whence of K is different from 2 and $f(t)$ has no multiple root in L. This completes the proof of our Theorem.

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