FIXED SUBALGEBRA OF A COMMUTATIVE FROBENIUS ALGEBRA

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Abstract. Let $B$ be a finite-dimensional commutative algebra generated by a single element, and let $A = B \otimes B$. We prove that the fixed subalgebra of $A$ under the involution $b_1 \otimes b_2 \mapsto b_2 \otimes b_1$ is Frobenius if and only if either the characteristic of $B$ is different from 2 or $B$ is separable.

1. Introduction. Let $A$ be a (finite-dimensional) Frobenius algebra over a field and $G$ a finite group of automorphisms of $A$. As is shown by J. L. Pascaud and J. Valette [3], it does not necessarily follow then that the fixed subalgebra $A^G$ under $G$ is Frobenius; indeed, they give an example of a local commutative Frobenius $A$ of any characteristic different from 2 and a group $G$ of order 2 such that $A^G$ is not Frobenius. (Note that for commutative algebras the concepts of “Frobenius” and “quasi-Frobenius” coincide.) In connection with this, N. Jacobson has raised the following question: Suppose $B$ is a commutative Frobenius algebra, or more particularly, an algebra generated by a single element, $A = B \otimes B$, and $G$ is the group of order 2 generated by the automorphism $\sigma: b_1 \otimes b_2 \mapsto b_2 \otimes b_1$ of $A$. Is then $A^G$ Frobenius too? In the present paper the question is answered affirmatively in the case where $B$ is singly generated and the characteristic of $B$ is different from 2.

2. Lemmas. Let $B$ be a finite-dimensional commutative algebra over a field $K$, $A = B \otimes_K B$ and $G = \{1, \sigma\}$, where $\sigma$ is the automorphism as above. Suppose that $w_1, w_2, \ldots, w_n$ are a basis of $B$ over $K$. Then the $n^2$ elements $w_i \otimes w_j$ ($1 \leq i, j \leq n$) form a basis of $A$. If we observe that $(w_i \otimes w_j)^\sigma = w_j \otimes w_i$ for all $i, j$, we can easily derive the following

Lemma 1. Let $w_1, w_2, \ldots, w_n$ be a basis of $B$ over $K$. Then the $n$ elements $w_i \otimes w_i$ ($1 \leq i < n$) and $n(n - 1)/2$ elements $w_i \otimes w_j + w_j \otimes w_i$ ($1 < i < j < n$) together form a basis of $A^G$ over $K$.

Let $L$ be an extension field of $K$. Then $B \otimes_K L$ is considered a commutative algebra over $L$, and we have $A \otimes_K L = (B \otimes_K L) \otimes_L (B \otimes_K L)$. The automorphism $\sigma$ of $A$ can also be regarded as an automorphism of $A \otimes_K L$ in the natural manner. Moreover, if $w_1, w_2, \ldots, w_n$ are a basis of $B$ over $K$ then it is also a basis

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of $B \otimes_K L$ over $L$, and therefore by applying Lemma 1 to $B \otimes_K L$ and to this basis we know that the $n + n(n - 1)/2 (= n(n + 1)/2)$ elements given in Lemma 1 together form a basis of $(A \otimes_K L)^G$ over $L$. Thus we have

**Lemma 2.** Let $L$ be an extension field of $K$. Then $(A \otimes_K L)^G = A^G \otimes_K L$.

Let $B = B_1 \oplus B_2 \oplus \cdots \oplus B_r$ be a decomposition of $B$ into a direct sum of orthogonal subalgebras. Put $A_{\alpha, \beta} = B_{\alpha} \otimes_K B_{\beta}$ for every $\alpha, \beta (= 1, 2, \ldots, r)$. Then $A_{\alpha, \beta}$'s are orthogonal subalgebras of $A$ such that $A = \sum A_{\alpha, \beta}$. Each $A_{\alpha, \beta}$ clearly satisfies $A_{\alpha, \beta}^G = A_{\alpha, \alpha}$, and in particular every $A_{\alpha, \alpha}$ is a $G$-invariant subalgebra. Let $(\alpha, \beta)$ be a pair such that $1 < \alpha < \beta < r$, and put $A_{\alpha, \beta}^G = A_{\alpha, \beta} \oplus A_{\beta, \alpha}$. Then $A_{\alpha, \beta}^G$ is also a $G$-invariant subalgebra, and we have $A_{\alpha, \beta}^G = \{a + a^\alpha | a \in A_{\alpha, \beta}\}$, since $A_{\alpha, \beta}$ and $A_{\beta, \alpha}$ are orthogonal; thus we know that, by associating $a \in A_{\alpha, \beta}$ with $a + a^\alpha$, $A_{\alpha, \beta}^G$ is isomorphic to $A_{\alpha, \beta}$ as algebras. We have therefore the following

**Lemma 3.** Let $B$ be a direct sum of orthogonal subalgebras $B_{\alpha}$, $1 < \alpha < r$. Let $A_{\alpha, \beta} = B_{\alpha} \otimes_K B_{\beta}$ for every $\alpha, \beta$, and let $A_{\alpha, \beta}^G = A_{\alpha, \beta} \oplus A_{\beta, \alpha}^G$ for every $\alpha, \beta$ such that $1 < \alpha < \beta < r$. Then the $r(r + 1)/2$ algebras $A_{\alpha, \alpha}$ and $A_{\alpha, \beta}$ are orthogonal $G$-invariant subalgebras of $A = B \otimes_K B$, and $A$ is a direct sum of these subalgebras; moreover, every algebra $A_{\alpha, \beta}^G$ is isomorphic to $A_{\alpha, \beta}$.

**Lemma 4.** Let $B$ be a commutative Frobenius algebra, and let $A = B \otimes_K B$. Then $A^G$ is Frobenius if and only if $A_{\alpha, \alpha}^G$ is Frobenius for every $\alpha$.

**Proof.** From Lemma 3 it follows that $A^G$ is the direct sum of orthogonal subalgebras $A_{\alpha, \alpha}^G$ and $A_{\alpha, \beta}^G$, where the latter algebras are isomorphic to $A_{\alpha, \beta}$. Therefore $A^G$ is Frobenius if and only if all $A_{\alpha, \alpha}^G$ and $A_{\alpha, \beta}$ are Frobenius. Since however $B$ is a Frobenius algebra, each direct summand subalgebra $B_{\alpha}$ and hence $A_{\alpha, \beta} = B_{\alpha} \otimes_K B_{\beta}$ are Frobenius algebras too. Therefore, in order that $A^G$ be Frobenius it is (necessary and) sufficient that $A_{\alpha, \alpha}^G$ be Frobenius for all $\alpha$.

3. **Main results.**

**Proposition 5.** Let $B$ be an algebra generated by a nonzero nilpotent element over a field $K$. Then $A^G$ is a Frobenius algebra if and only if the characteristic of $K$ is different from 2.

**Proof.** Let $B = K[c]$ with a nilpotent element $c \neq 0$ of $B$. Let $n$ be the least positive integer such that $c^n = 0$. Then $n > 2$, and the $n$ elements $1, c, c^2, \ldots, c^{n-1}$ form a basis of $B$ over $K$. Therefore, by Lemma 1, the $n$ elements $c^i \otimes c^j (0 \leq i \leq n - 1)$ and $n(n - 1)/2$ elements $c^i \otimes c^j + c^j \otimes c^i (0 < i < j \leq n - 1)$ together form a basis of $A^G$ over $K$. If we remove $1 \otimes 1$ (the unit element of $A^G$) out of the basis then we have a subset $W$. Let $N$ be the $K$-subspace of $A^G$ generated by $W$. Then since every element of $W$ is nilpotent, $N$ is contained in the radical of $A^G$. It is however clear that $A^G = N \oplus K(1 \otimes 1)$, and this, together with the fact that the radical is a proper ideal of $A^G$, implies that $N$ coincides with the radical and the factor algebra $A^G/N$ is isomorphic to $K$. Thus, $A^G$ is a local ring and has an up-to-isomorphism unique simple module, which is one-dimensional over $K$. Moreover, $N$ is the ideal of $A^G$ generated by $c \otimes c$ and
1 \otimes c + c \otimes 1$, because 
\[ c^i \otimes c^i = (c \otimes c)^i \text{ for } i > 0, \]
\[ 1 \otimes c^2 + c^2 \otimes 1 = (1 \otimes c + c \otimes 1)^2 - 2(c \otimes c), \]
\[ 1 \otimes c^i + c^i \otimes 1 = (1 \otimes c + c \otimes 1)(1 \otimes c^{i-1} + c^{i-1} \otimes 1) \]
\[ - (c \otimes c)(1 \otimes c^{i-2} + c^{i-2} \otimes 1) \text{ for } i > 2 \]
and
\[ c^i \otimes c^j + c^j \otimes c^i = (c \otimes c)^j(1 \otimes c^{j-i} + c^{j-i} \otimes 1) \text{ for } j > i > 0. \]

(More precisely, by using these equalities, it is possible to show without difficulty that every element of \( W \) is expressed as a polynomial of \( c \otimes c \) and \( 1 \otimes c + c \otimes 1 \) with integral coefficients and without constant term.)

Let now \( M \) be the socle of \( A^G \), i.e., the sum of all simple ideals of \( A^G \). As is well known, \( M \) is then the annihilator of the radical \( N \) in \( A^G \) and therefore is the annihilator of the generators \( c \otimes c \) and \( 1 \otimes c + c \otimes 1 \) of \( N \) in \( A^G \). Let \( M' \) be the annihilator of \( c \otimes c \) in \( A^G \). If we observe that
\[ (c \otimes c)(c^i \otimes c^j) = c^{i+1} \otimes c^{j+1} \]
is a member of \( W \) or \( = 0 \) according as \( 0 < i < n - 2 \) or \( j = n - 1 \) and also
\[ (c \otimes c)(c^i \otimes c^j + c^j \otimes c^i) = c^{i+1} \otimes c^{j+1} + c^{j+1} \otimes c^{i+1} \]
is a member of \( W \) or \( = 0 \) according as \( 0 < i < j < n - 2 \) or \( j = n - 1 \), then we know that \( M' \) is the \( K \)-subspace of \( A^G \) generated by \( c^{n-1} \otimes c^{n-1}, 1 \otimes c^{n-1} + c^{n-1} \otimes 1, c \otimes c^{n-1} + c^{n-1} \otimes c, \ldots, c^{n-2} \otimes c^{n-1} + c^{n-1} \otimes c^{n-2} \). Next multiply these \( n \) elements by \( 1 \otimes c + c \otimes 1 \). Then we have
\[ (1 \otimes c + c \otimes 1)(c^{n-1} \otimes c^{n-1}) = 0, \]
\[ (1 \otimes c + c \otimes 1)(1 \otimes c^{n-1} + c^{n-1} \otimes 1) = c \otimes c^{n-1} + c^{n-1} \otimes c, \ldots, \]
\[ (1 \otimes c + c \otimes 1)(c^{n-3} \otimes c^{n-1} + c^{n-1} \otimes c^{n-3}) = c^{n-2} \otimes c^{n-1} + c^{n-1} \otimes c^{n-2}, \]
\[ (1 \otimes c + c \otimes 1)(c^{n-2} \otimes c^{n-1} + c^{n-1} \otimes c^{n-2}) = 2(c^{n-2} \otimes c^{n-1}), \]
where the last element is \( 0 \) if and only if the characteristic of \( K \) is \( 2 \). This means that the annihilator of \( 1 \otimes c + c \otimes 1 \) in \( M' \), i.e., the annihilator \( M \) of \( N \) in \( A^G \) is \( K \cdot (c^{n-1} \otimes c^{n-1}) \) or \( K \cdot (c^{n-1} \otimes c^{n-1}) \oplus K \cdot (c^{n-2} \otimes c^{n-1} + c^{n-1} \otimes c^{n-2}) \) according as the characteristic of \( K \) is different from 2 or equal to 2. Now, since \( A^G \) is a local ring, \( A^G \) is a Frobenius algebra if and only if \( A^G \) has a unique simple ideal, or equivalently, its socle \( M \) is simple. But since \( A^G / N \) is isomorphic to \( K \), this is equivalent to the condition that \( M \) is one-dimensional over \( K \). Thus we have that \( A^G \) is Frobenius if and only if \( M = K \cdot (c^{n-1} \otimes c^{n-1}), \) i.e., the characteristic of \( K \) is different from 2.

**Corollary 6.** Let \( B = K[a] \) and let the minimal polynomial \( f(t) \) of \( a \) over \( K \) be a power of a linear polynomial in \( K \). Then \( A^G \) is a Frobenius algebra if and only if the characteristic of \( K \) is different from 2 or \( f(t) \) is linear.

**Proof.** Let \( f(t) = (t - \lambda)^n, \lambda \in K \). If we put \( c = a - \lambda \) then \( B = K[c] \) and \( n \) is the least positive integer such that \( c^n = 0 \). Thus, if \( n > 1 \) then \( c \neq 0 \) and by Proposition 5, \( A^G \) is Frobenius if and only if the characteristic of \( K \) is different
from 2, while if \( n = 1 \) then \( c = 0 \) and clearly \( B = K = A = A^G \) and the \( K \)-algebra \( K \) is trivially a Frobenius algebra.

**Theorem 7.** Let \( B \) be \( K[a] \) and let \( f(t) \) be the minimal polynomial of \( a \) over \( K \). Let \( A = B \otimes_K B \) and let \( G = \{1, \sigma\} \), where \( \sigma \) is the automorphism of \( A \) defined by \((b_1 \otimes b_2)^\sigma = b_2 \otimes b_1 \) for \( b_1, b_2 \in B \). Then \( A^G \) is a Frobenius algebra over \( K \) if and only if the characteristic of \( K \) is different from 2 or \( f(t) \) is separable (i.e., \( f(t) \) has no multiple root in a splitting field).

**Proof.** (i) Assume first that \( K \) is algebraically closed. Then \( f(t) \) is decomposed as
\[
f(t) = (t - \lambda_1)^{n_1}(t - \lambda_2)^{n_2} \cdots (t - \lambda_r)^{n_r},
\]
where \( \lambda_1, \lambda_2, \ldots, \lambda_r \) are distinct roots of \( f(t) \) in \( K \). As is well known, this decomposition yields a direct decomposition \( B = B_1 \oplus B_2 \oplus \cdots \oplus B_r \) into orthogonal subalgebras such that if \( a_a \) is the \( B_a \)-component of \( a \) then \( B_a = K[a_a] \) and \( (t - \lambda_a)^{n_a} \) is the minimal polynomial of \( a_a \) over \( K \) for every \( a \). Now, as a homomorphic image of the polynomial ring \( K[t] \), \( B \) is a principal ideal ring and therefore a Frobenius algebra (cf. Nakayama [1, Theorem 16]). From this follows by Lemma 4 that \( A^G \) is Frobenius if and only if \( A_{a,a}^G \), where \( A_{a,a} = B_a \otimes_K B_a \), is Frobenius for every \( a \). On the other hand, according to Corollary 6, each \( A_{a,a}^G \) is Frobenius if and only if the characteristic of \( K \) is different from 2 or \( n_a = 1 \). Thus we have that \( A^G \) is a Frobenius algebra if and only if either the characteristic of \( K \) is different from 2 or \( n_1 = n_2 = \cdots = n_r = 1 \).

(ii) Suppose that \( K \) is not necessarily algebraically closed. Let \( L \) be the algebraic closure of \( K \). Then \((A \otimes_K L)^G = A^G \otimes_K L \) by Lemma 2. Therefore it follows by Nakayama [1, Theorem 14] or Nakayama and Nesbitt [2, Theorem 5] that \( A^G \) is Frobenius over \( K \) if and only if \((A \otimes_K L)^G \) is Frobenius over \( L \). But if we observe that \( A \otimes_K L = (B \otimes_K L) \otimes_L (B \otimes_K L) \), \( B \otimes_K L = L[a] \) and the minimal polynomial of \( a \) over \( L \) is the same as \( f(t) \), we can conclude by the above case (i) that the latter condition is equivalent to the condition that the characteristic of \( L \) whence of \( K \) is different from 2 and \( f(t) \) has no multiple root in \( L \). This completes the proof of our Theorem.

**References**


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