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A NONLINEAR ERGODIC THEOREM FOR AN AMENABLE SEMIGROUP OF NONEXPANSIVE MAPPINGS IN A HILBERT SPACE

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ABSTRACT. We prove a nonlinear ergodic theorem for noncommutative semigroups of nonexpansive mappings in a Hilbert space. Furthermore, we give a necessary and sufficient condition for a noncommutative semigroup to have a fixed point.

1. Introduction. Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and C a nonempty closed convex subset of H. A mapping $T\colon C\to C$ is called nonexpansive on C, or $T\in \mathrm{Cont}(C)$ if

$$\|Tx-Ty\|\leq\|x-y\| \quad \text{for every } x,y\in C.$$

Let $F(T)$ be the set of fixed points of T, that is, $F(T)=\{z\in C\colon Tz=z\}$. Then, the set $F(T)$ is obviously closed and convex. Let $S=\{S(t)\colon t\geq 0\}$ be a family of nonexpansive mappings of C into itself such that $S(0)=I$, $S(t+s)=S(t)S(s)$ for all $t,s\in[0,\infty)$ and $S(t)x$ is continuous in $t\in[0,\infty)$ for each $x\in C$. Then, S is called a nonexpansive semigroup on C. The fixed point set $F(S)$ of S is defined by

$$F(S)=\{x\in C\colon S(t)x=x \text{ for all } t\in[0,\infty)\}.$$

The first nonlinear ergodic theorem for nonexpansive mappings was established by Baillon [1]: Let $C\subset H$, $T\in\mathrm{Cont}(C)$ and $F(T)\neq\varnothing$. Then the Cesàro means

$$S_n(x)=\frac{1}{n}\sum_{k=0}^{n-1}T^kx$$

converge weakly as $n\to+\infty$ to a fixed point of T for each $x\in C$. A corresponding result for nonexpansive semigroups on C was given by Baillon [2] and Baillon-Brézis [3]. Nonlinear ergodic theorems for general commutative semigroups of nonexpansive mappings were given by Brézis-Browder [4] and Hirano-Takahashi [6].

In this paper, we prove a nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings of C into itself. Furthermore we obtain a necessary and sufficient condition for a left amenable semigroup of nonexpansive mappings to have a common fixed point. This is a generalization of Pazy's results [7] and [9].

2. Nonlinear ergodic theorem. Let S be an abstract semigroup and $m(S)$ the Banach space of all bounded real valued functions on S with the supremum norm. For each $s\in S$ and $f\in m(S)$, we define elements f_s and f^s in $m(S)$ given by

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$f_s(t) = f(st)$ and $f^s(t) = f(ts)$ for all $t \in S$. An element $\mu \in m(S)^*$ (the dual space of $m(S)$) is called a mean on S if $\|\mu\| = \mu(1) = 1$. A mean μ is called left (right) invariant if $\mu(f_s) = \mu(f)$ ($\mu(f^s) = \mu(f)$) for all $f \in m(S)$ and $s \in S$. An invariant mean is a left and right invariant mean. A semigroup which has a left (right) invariant mean is called left (right) amenable. A semigroup which has an invariant mean is called amenable. Day [5] proved that a commutative semigroup is amenable. We also know that $\mu \in m(S)^*$ is a mean on S if and only if

$$\inf\{f(s): s \in S\} \leqslant \mu(f) \leqslant \sup\{f(s): s \in S\}$$

for every $f \in m(S)$.

Now we prove a nonlinear ergodic theorem for noncommutative semigroups of nonexpansive mappings in a Hilbert space. The proof employs the methods of [8], [11] and [12].

THEOREM 1. *Let C be a nonempty closed convex subset of a real Hilbert space H and S be an amenable semigroup of nonexpansive mappings t of C into itself. Suppose that*

$$F(S) = \bigcap \{F(t): t \in S\} \neq \varnothing.$$

Then, there exists a nonexpansive retraction P of C onto $F(S)$ such that $Pt = tP = P$ for every $t \in S$ and $Px \in \overline{\mathrm{co}}\{tx: t \in S\}$ for every $x \in C$, where $\overline{\mathrm{co}}\, A$ is the closure of the convex hull of A.

PROOF. Let μ be an invariant mean on S and $x \in C$. Then, since $F(S) \neq \varnothing$, $\{tx: t \in S\}$ is bounded and hence, for each y in H, the real-valued function $t \to \langle tx, y\rangle$ is in $m(S)$. Denote by $\mu_t\langle tx, y\rangle$ the value of μ at this function. By linearity of μ and of the inner product, this is linear in y; moreover, since

$$|\mu_t\langle tx, y\rangle| \leqslant \|\mu\| \cdot \sup_t |\langle tx, y\rangle| \leqslant \left(\sup_t \|tx\|\right) \cdot \|y\|,$$

it is continuous in y, so by the Riesz theorem, there exists an $x_0 \in H$ such that $\mu_t\langle tx, y\rangle = \langle x_0, y\rangle$ for every $y \in H$. Setting $Px = x_0$, we have

$$Px \in \overline{\mathrm{co}}\,\{tx: t \in S\}.$$

In fact, if $Px \notin \overline{\mathrm{co}}\{tx: t \in S\}$, then by the separation theorem there exists a $y_0 \in H$ such that

$$\langle Px, y_0\rangle < \inf\{\langle z, y_0\rangle: z \in \overline{\mathrm{co}}\,\{tx: t \in S\}\}.$$

So, we have

$$\begin{aligned}\inf_t \langle tx, y_0\rangle \leqslant \mu_t\langle tx, y_0\rangle &= \langle Px, y_0\rangle \\ &< \inf\{\langle z, y_0\rangle: z \in \overline{\mathrm{co}}\,\{tx: t \in S\}\} \\ &\leqslant \inf_t \langle tx, y_0\rangle.\end{aligned}$$

This is a contradiction. Let $s \in S$. Then we have

$$\begin{aligned}0 &\leqslant \|tx - x_0\|^2 - \|stx - sx_0\|^2 \\ &\leqslant \|tx - sx_0\|^2 + 2\langle tx - sx_0, sx_0 - x_0\rangle \\ &\quad + \|sx_0 - x_0\|^2 - \|stx - sx_0\|^2\end{aligned}$$

and hence

$$\begin{aligned}
0 &\leqslant \mu_t\left(\|tx - sx_0\|^2 + 2\langle tx - sx_0, sx_0 - x_0\rangle + \|sx_0 - x_0\|^2 - \|stx - sx_0\|^2\right) \\
&= \mu_t\|tx - sx_0\|^2 + 2\langle x_0 - sx_0, sx_0 - x_0\rangle \\
&\quad + \|sx_0 - x_0\|^2 - \mu_t\|tx - sx_0\|^2 \\
&= 2\langle x_0 - sx_0, sx_0 - x_0\rangle + \|sx_0 - x_0\|^2 \\
&= -\|x_0 - sx_0\|^2.
\end{aligned}$$

This implies $sx_0 = x_0$ for every $s \in S$ and hence we have $sPx = Px$ for every $s \in S$. From

$$\langle Psx, y\rangle = \mu_t\langle tsx, y\rangle = \mu_t\langle tx, y\rangle = \langle Px, y\rangle$$

and

$$\langle P^2x, y\rangle = \mu_t\langle tPx, y\rangle = \mu_t\langle Px, y\rangle = \langle Px, y\rangle,$$

it follows that $Ps = P$ for every $s \in S$ and $P^2 = P$. At last, we prove that P is nonexpansive. In fact, we have

$$\begin{aligned}
\|Px - Py\|^2 = \langle Px - Py, Px - Py\rangle &= \mu_t\langle tx - ty, Px - Py\rangle \\
&\leqslant \left(\sup_t \|tx - ty\|\right)\cdot\|Px - Py\| \\
&\leqslant \|x - y\|\cdot\|Px - Py\|
\end{aligned}$$

for every $x, y \in C$.

As a direct consequence, we have

COROLLARY 1. *Let C be a nonempty closed convex subset of a real Hilbert space H and S be a commutative semigroup of nonexpansive mappings t of C into itself. Suppose that $F(S) \neq \varnothing$. Then there exists a nonexpansive retraction P of C onto $F(S)$ such that $Pt = tP = P$ for every $t \in S$ and $Px \in \overline{\mathrm{co}}\{tx\colon t \in S\}$ for every $x \in C$.*

By the method of Theorem 1, we can prove the following

THEOREM 2. *Let C be a nonempty closed convex subset of a real Hilbert space H and S be a left amenable semigroup of nonexpansive mappings t of C into itself. Then, $F(S) \neq \varnothing$ if and only if there exists an $x_0 \in C$ such that $\{tx_0\colon t \in S\}$ is bounded.*

As direct consequences, we obtain Pazy's results [7] and [9].

COROLLARY 2. *Let C be a nonempty closed convex subset of a real Hilbert space H and T be a nonexpansive mapping of C into itself. Then, $F(T) \neq \varnothing$ if and only if there exists an element $x_0 \in C$ such that the sequence $\{T^n x_0\colon n = 1, 2, \ldots\}$ is bounded.*

COROLLARY 3. *Let C be a nonempty closed convex subset of a real Hilbert space H and $S = \{S(t)\colon t \geqslant 0\}$ be a nonexpansive semigroup on C. Then, $F(S) \neq \varnothing$ if and only if there exists an element $x_0 \in C$ such that $\{S(t)x_0\colon t \geqslant 0\}$ is bounded.*

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